Multifractal Background of Monofractal Finite Signals with Long Memory

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Outline

1. Fractality and Multifractality – general overview
2. Generalized Hurst exponents vs. Hölder description
3. Fourier transform based method producing stationary time series with long–memory
4. Searching for multifractal noise in monofractal signals with long memory – numerical results & analytic fit
5. Discussion and conclusions

Multifractality

- Monofractality

Self-similar objects, scaling properties do not depend on particular scale

- Multifractality

Self-similar objects, scaling properties are different for various scales

Multifractality

- Monofractality

Self-similar objects, scaling properties do not depend on particular scale

In terms of TS

The same scaling properties for all time scales
(e.g. seconds, minutes, hours)

- Multifractality

Self-similar objects, scaling properties are different for various scales

In terms of TS

Different scaling properties for various time scales

Multifractality

$$t: \ \ldots < S_x^{(i)} < S_x^{(i+1)} < \ldots < S_x^{(j)} < \ldots$$

fractal properties for $S_x^{(i)} < t < S_x^{(i+1)}$ vary with i

MF-DFA most effective tool to search for multifractality in time series [J. Kantelhardt, et.al. Phys.A **316** (2002) 87-114]

$\{X(i)\}$ – time series, τ – size of non-overlapping boxes,
N_τ – # of boxes

$$F_q(\tau) = \left\{ \frac{1}{2N_\tau} \sum_{i=1}^{2N_\tau} \left[F^2(i,\tau) \right]^{\frac{q}{2}} \right\}^{\frac{1}{q}} \qquad q \neq 0$$

$$F_0(\tau) = \left\{ \frac{1}{4N_\tau} \sum_{i=1}^{2N_\tau} \ln \left[F^2(i,\tau) \right] \right\} \qquad q = 0$$

Multifractality

$$t: \ldots < S_x^{(i)} < S_x^{(i+1)} < \ldots < S_x^{(j)} < \ldots$$

fractal properties for $S_x^{(i)} < t < S_x^{(i+1)}$ vary with i

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$\{X(i)\}$ – time series, τ – size of non-overlapping boxes,
N_τ – # of boxes

where

$$F^2(i,\tau) = \frac{1}{\tau}\sum_{k=1}^{\tau}\left\{X\left[N-(i-N_\tau)\tau+k\right]-\tilde{X}_i(k)\right\}^2$$

Translation between Hölder and Hurst description of multifractality

Singularity spectrum $f(\alpha)$ is derived as Legandre transform of classical multifractal scaling exponents $\tau(q) = qh(q) - 1$.

$$\alpha(q) = \tau'(q) = h(q) + qh'(q)$$

$$f(\alpha) = q\alpha - \tau(\alpha) = q[\alpha - h(q)] + 1$$

In this description $\alpha(q)$ is called the Hölder exponent and $f(\alpha)$ is its spectrum.

Generalized Hurst and Hölder descriptions for real data

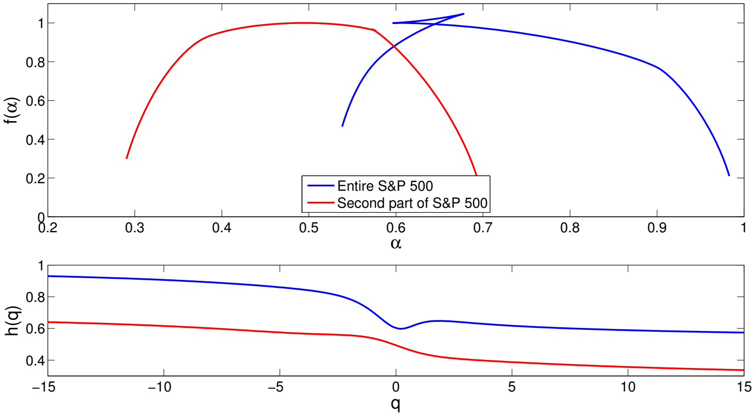

Figure: Example of Hurst and Hölder descriptions for entire S&P500 index and its second part (since 1997 till now) [Ł. Czarnecki, D. Grech Acta Phys.Pol. A **117** (2010) 4]

Fourier Filtering Method (FFM)

Algorithm producing long memory correlated time series
[C.-K. Peng, et.al. Phys.Rev.A **44**, 2239 (1991)]

$$C(\ell) = \langle x_i x_{i+l} \rangle \sim \ell^{-\gamma}, \quad \gamma \in [0,1] \quad \gamma = 2 - 2H$$

1. produce a stationary sequence $\xi_i, \; i = 1, \ldots, L$ uncorrelated random numbers drawn from $\mathcal{N}(0, \sigma)$ distribution
2. calculate Fourier transform of generated data
$$\tilde{\xi}_q = \sum_{k=0}^{L-1} \xi_k e^{-2\pi i \frac{qk}{L}}$$
3. apply filter function $S(q) = q^{\gamma - 1}$
$$\tilde{x}_q = \sqrt{S(q)}\, \tilde{\xi}_q$$
4. calculate inverse Fourier transform of filtered sequence $\tilde{x}_q$ to obtain time series with desired long range correlation (γ)

Example

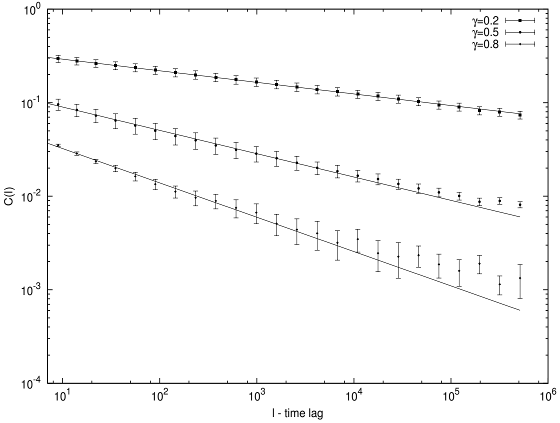

Figure: The test of long memory for artificially constructed (FFM) time series

Real time series

characteristics

Real time series

characteristics

- finite size

Real time series
characteristics

- finite size
- long memory

Real time series

characteristics

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How much do they affect multifractal structure of TS ?

Real time series

characteristics

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How much do they affect multifractal structure of TS ?

Do they produce multifractal structure ('multifractal noise') in monofractal signals ?

Real time series

characteristics

- finite size
- long memory

How much do they affect multifractal structure of TS ?

Do they produce multifractal structure ('multifractal noise') in monofractal signals ?

$$\Delta h(\gamma, L) \qquad \Delta\alpha(\gamma, L)$$

Investigated time series

Monofractal time series with long memory generated with FFM

$$L = 2^9,\ 2^{10},\ 2^{12},\ 2^{14},\ 2^{16},\ 2^{18},\ 2^{19},\ 2^{20}$$

$$\gamma = 0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9$$

Investigated time series

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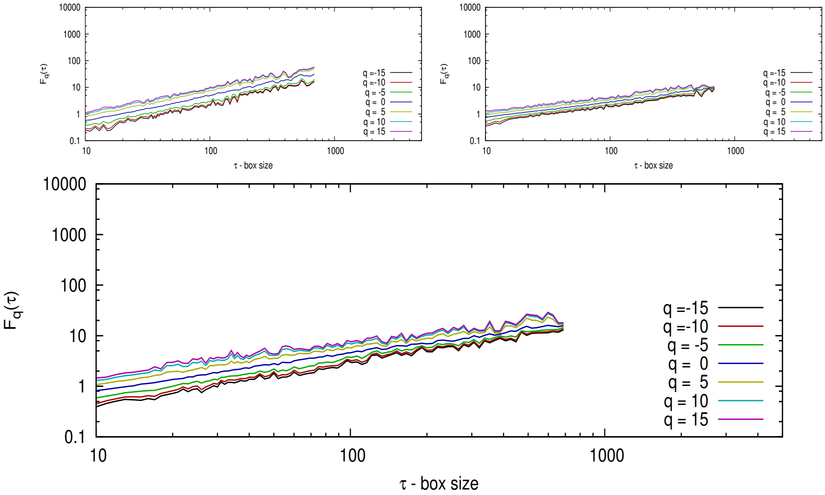
$$L = 2^9,\ 2^{10},\ 2^{12},\ 2^{14},\ 2^{16},\ 2^{18},\ 2^{19},\ 2^{20}$$

$$\gamma = 0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9$$

Statistical ensemble of 100 series for each parameter pair $(\gamma,\ L)$ was studied

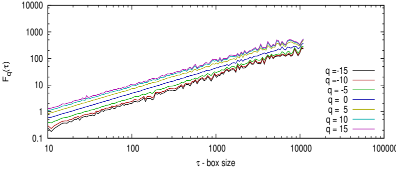
Scaling range for MFDFA method

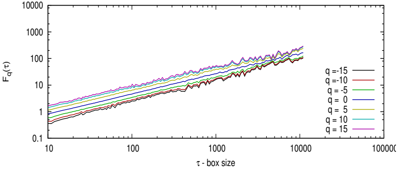
$L = 2^{12} = 4096$

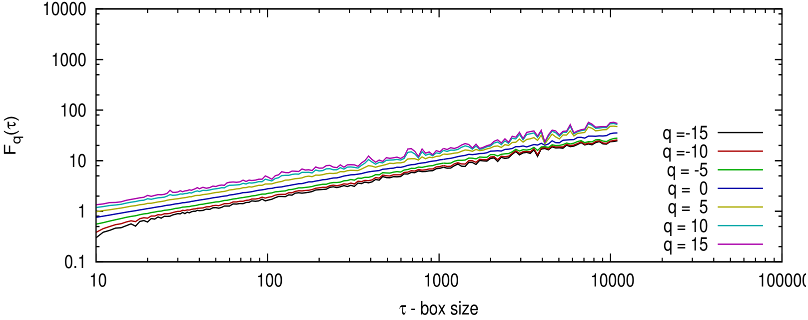


Scaling range for MFDFA method

$L = 2^{16} = 65536$

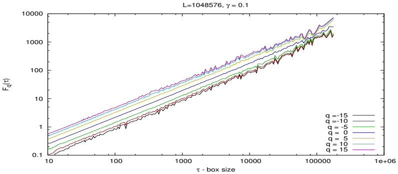


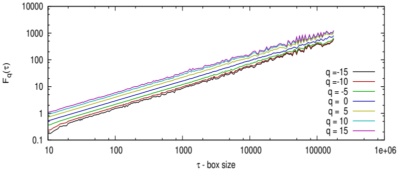


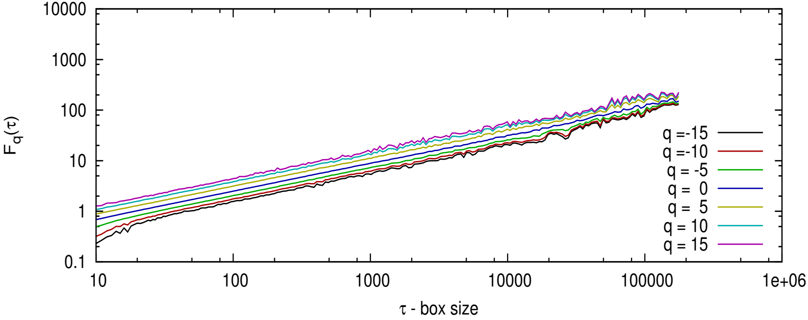


Scaling range for MFDFA method

$L = 2^{20} = 1048576$

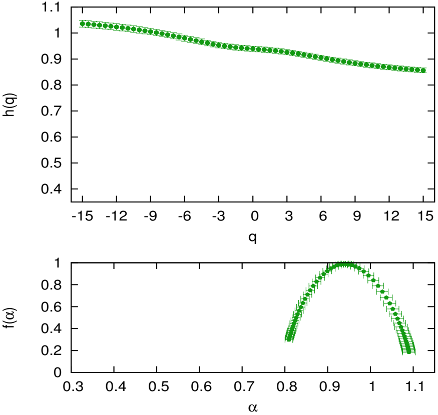




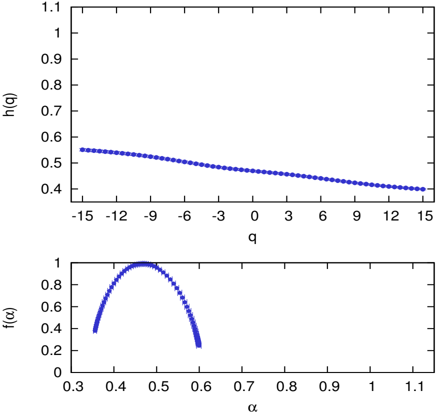


Generalized Hurst exponent and Hölder exponent spectrum

$L = 2^{12} = 4096, \quad \gamma = 0.1$ (monofractal)



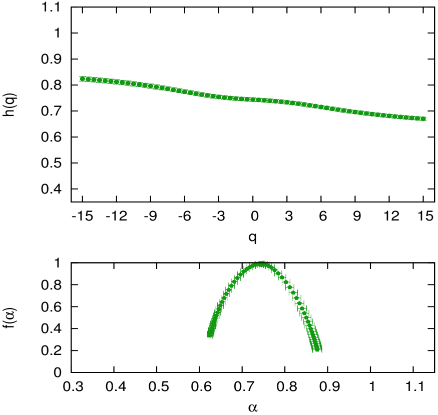
Correlated



Shuffled

Generalized Hurst exponent and Hölder exponent spectrum

$L = 2^{12} = 4096, \quad \gamma = 0.5$ (monofractal)



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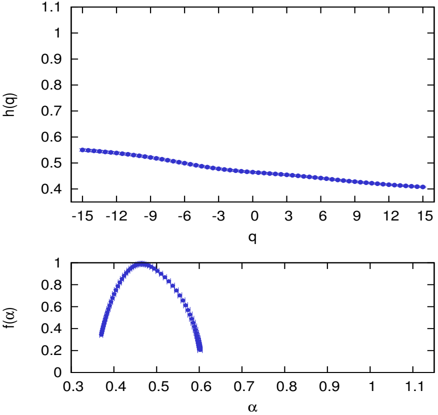
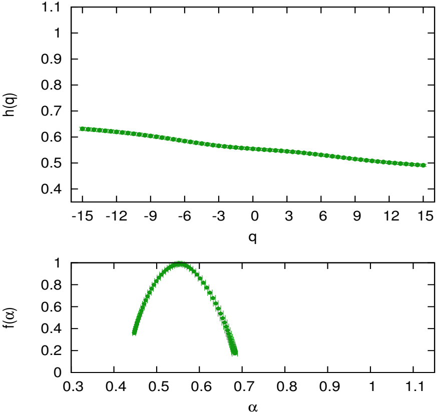


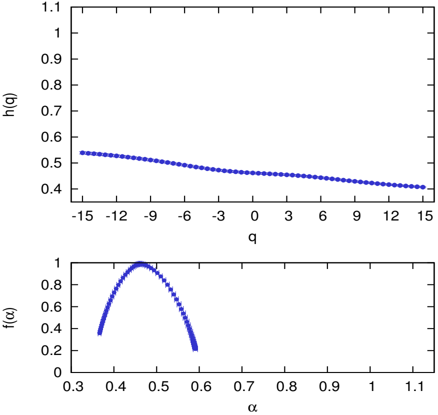
Shuffled

Generalized Hurst exponent and Hölder exponent spectrum

$L = 2^{12} = 4096, \quad \gamma = 0.9$ (monofractal)



Correlated



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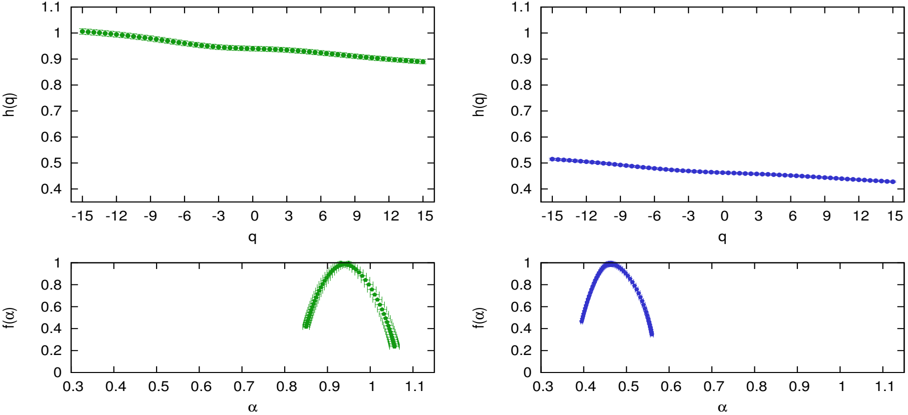

Correlated | Shuffled

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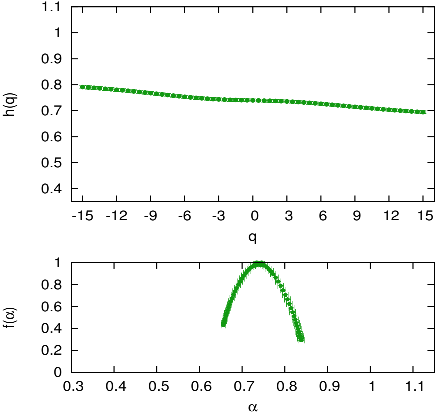
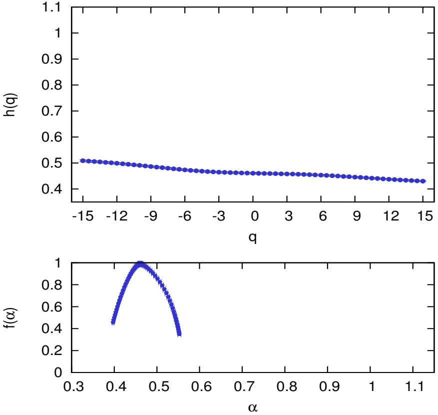


Correlated



Shuffled

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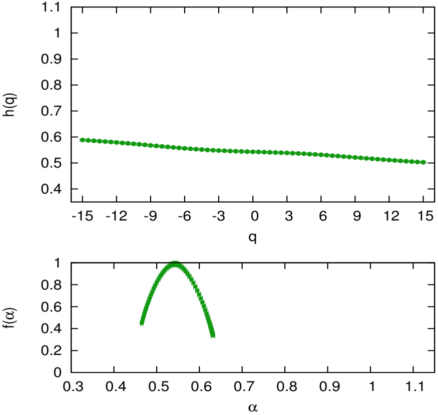

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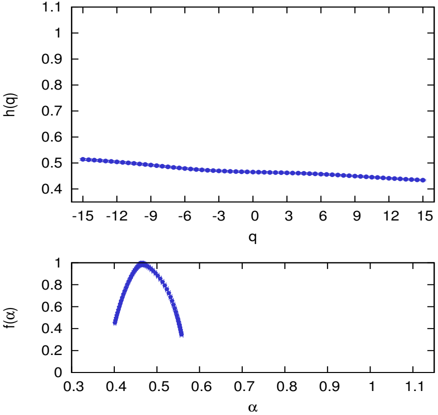
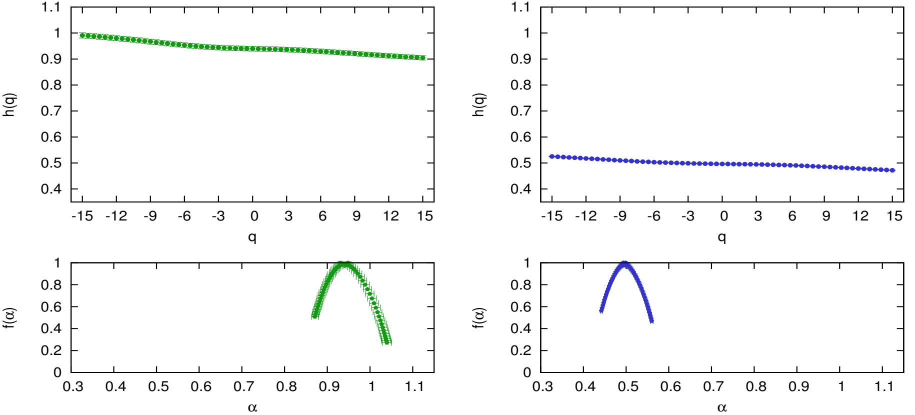


Shuffled

Generalized Hurst exponent and Hölder exponent spectrum

$L = 2^{20} = 1048576, \quad \gamma = 0.1$ (monofractal)



Correlated Shuffled

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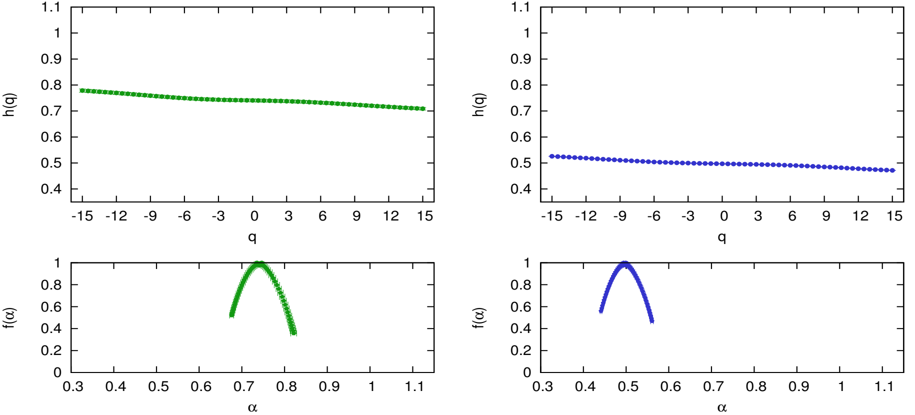

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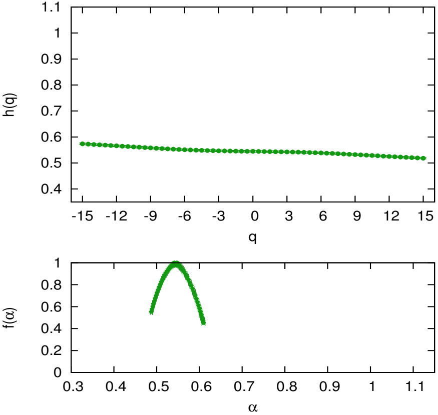

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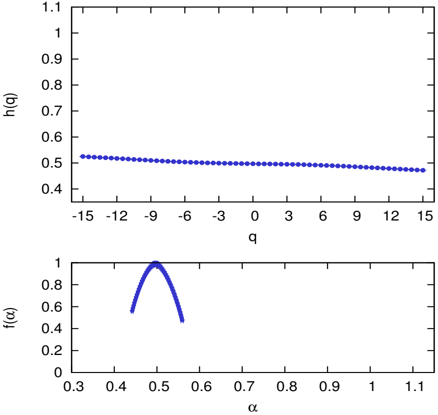

Shuffled

Finite size effects

Hurst description, $L = 2^{12} = 4096$

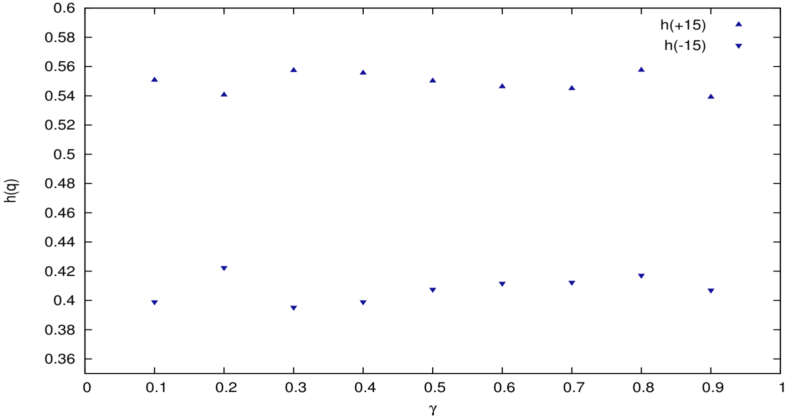

Figure: Maximal and minimal value of generalized Hurst parameter

Finite size effects

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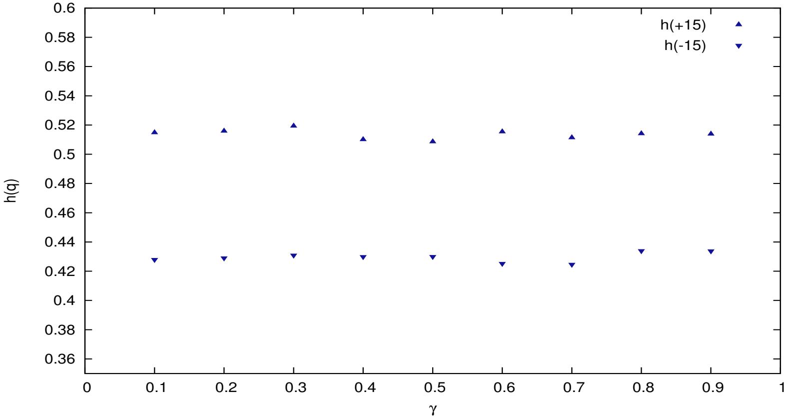

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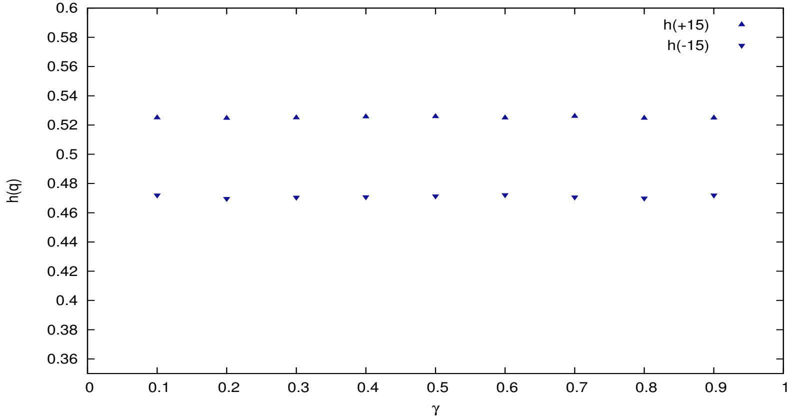

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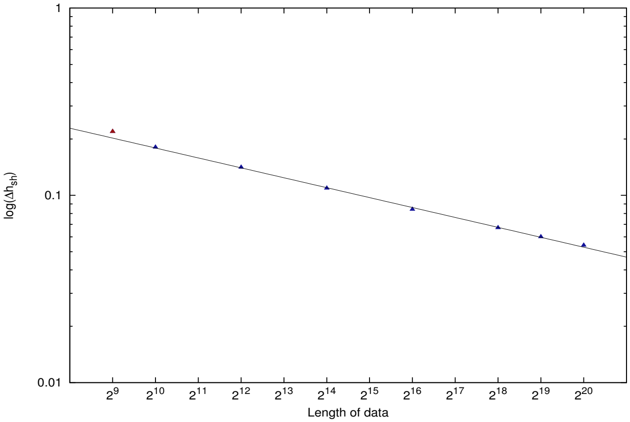


Figure: Finite size effects revealing multifractality in monofractal signals

Finite size effects

Hurst description

$$\Delta h_{sh}(L) = C_{sh} L^{-\eta_{sh}}$$

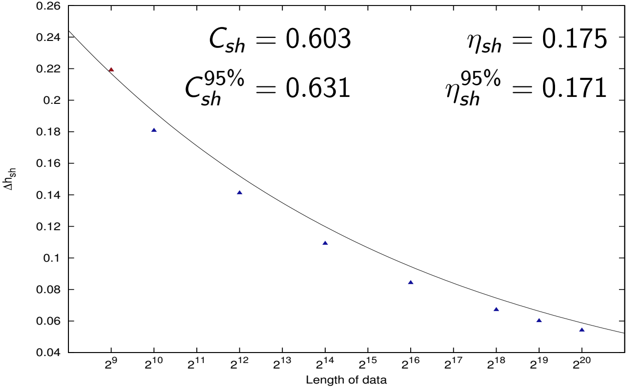


Figure: 95% confidence level for multifractal effects of finite shuffled signals

Finite size effects

Hölder description

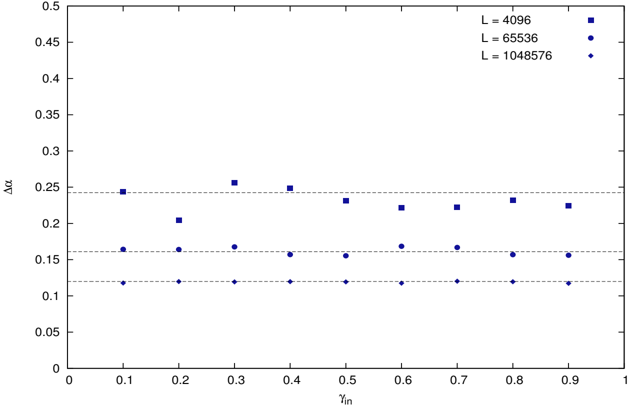


Figure: Singularity spectrum width $\Delta\alpha(L)$ for shuffled signals with long memory

Finite size effects

Hölder description

$$\Delta\alpha_{sh}(L) = D_{sh} L^{-\zeta_{sh}}$$

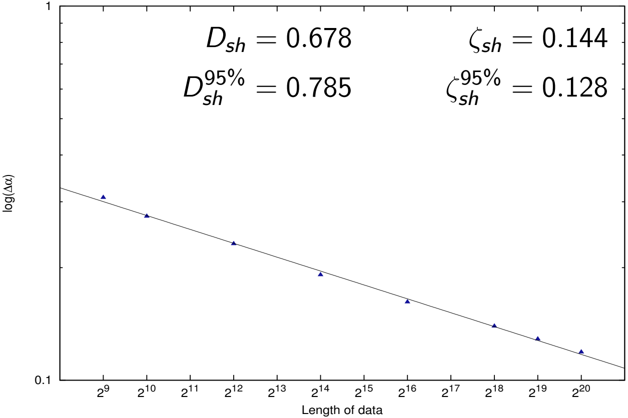


Figure: Singularity spectrum width as a function of TS length

Long memory effects

Hölder description

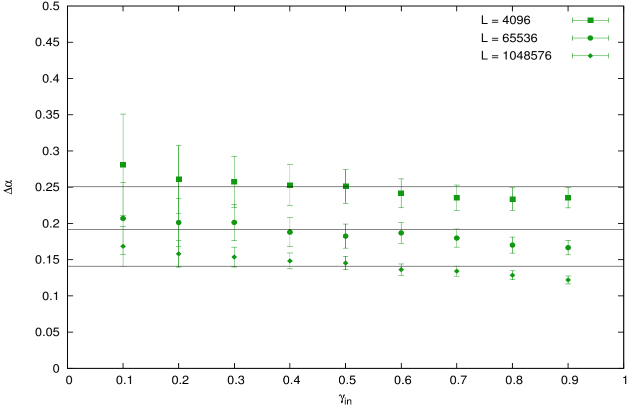


Figure: Dependence between singularity spectrum width and residual long memory effect on multifractal spectrum width

Long memory effects

Hölder description, $\gamma = 0.1$

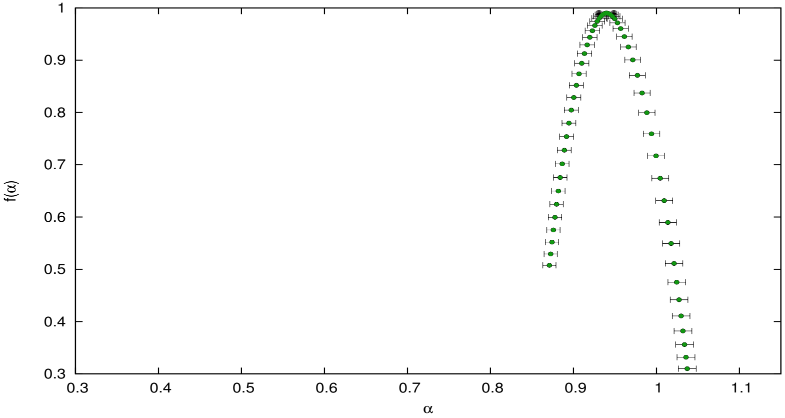


Figure: Moving multifractal spectrum for finite signals with long memory

Long memory effects

Hölder description, $\gamma = 0.5$

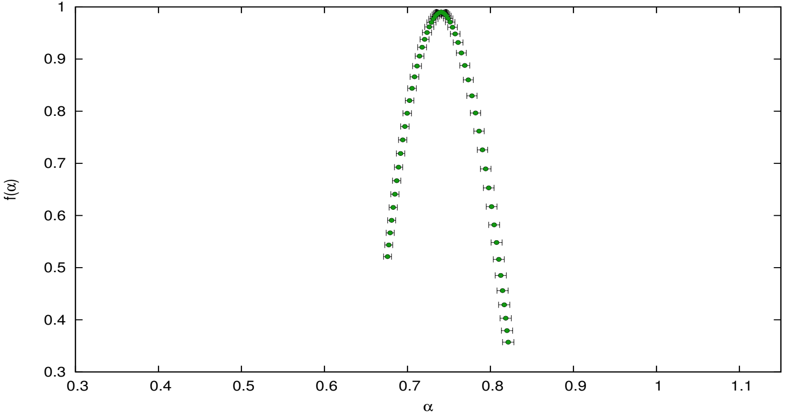


Figure: Moving multifractal spectrum for finite signals with long memory

Long memory effects

Hölder description, $\gamma = 0.9$

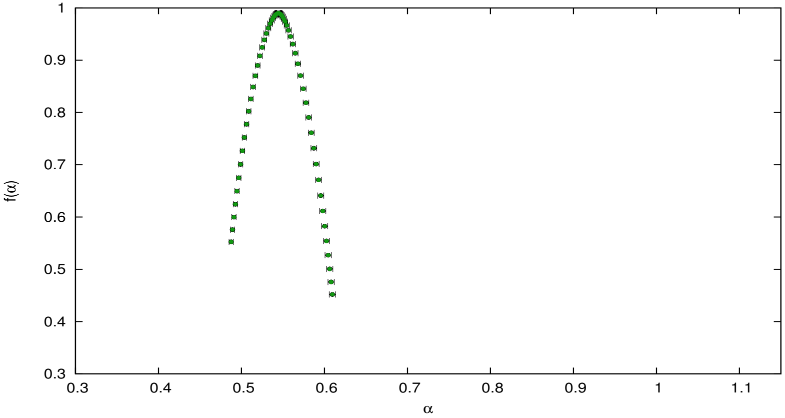


Figure: Moving multifractal spectrum for finite signals with long memory

Long memory effects

Hölder description, $L = 2^{12} = 4096$

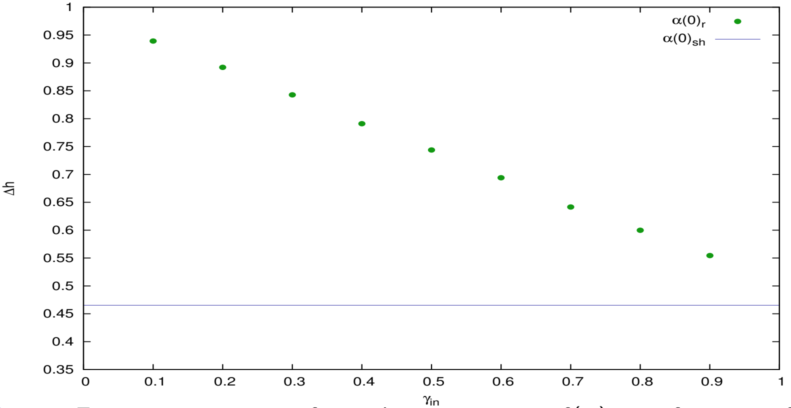
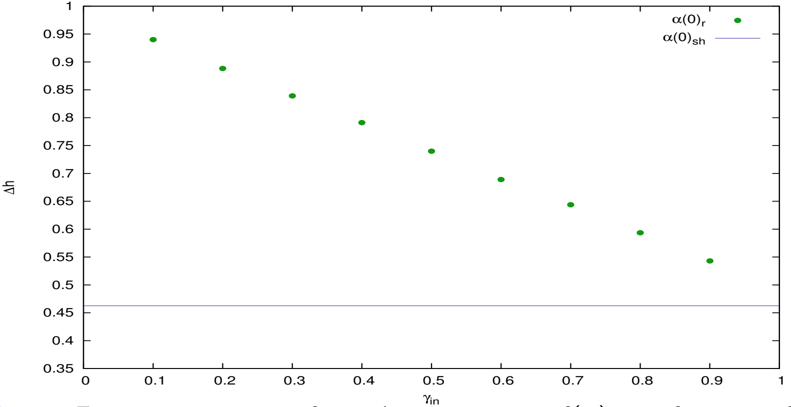


Figure: Extremum position of singularity spectrum $f(\alpha)$ as a function of autocorrelation exponent γ

Long memory effects

Hölder description, $L = 2^{16} = 65536$



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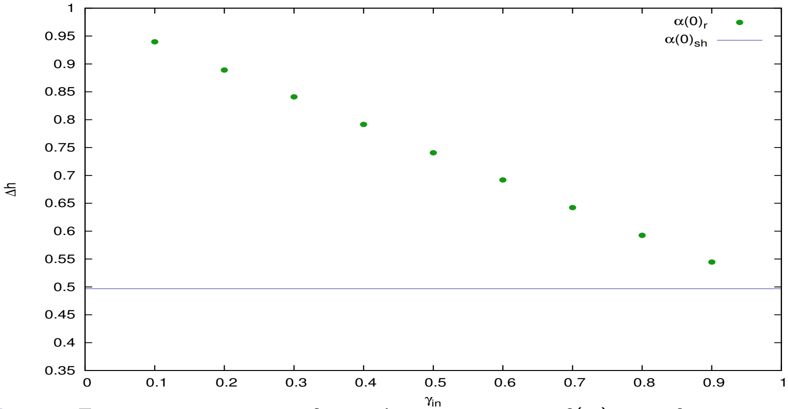
Hölder description, $L = 2^{20} = 1048576$



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Long memory effects

Hurst description, $L = 2^{20} = 1048576$

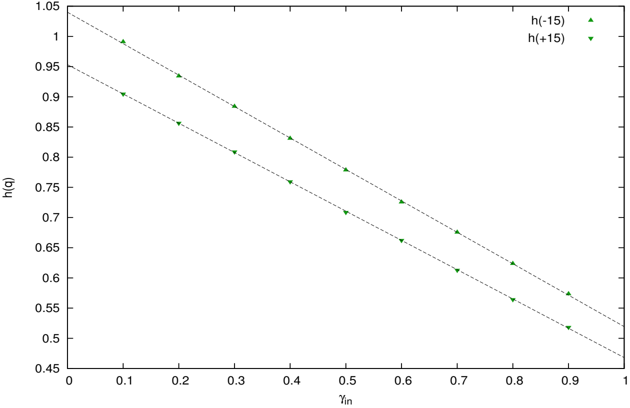

Figure: The edge values of generalized Hurst exponents for long memory correlated monofractal signal

Long memory effects

Hurst description

$$\Delta h_0(L) = C_0 L^{-\eta_0}$$

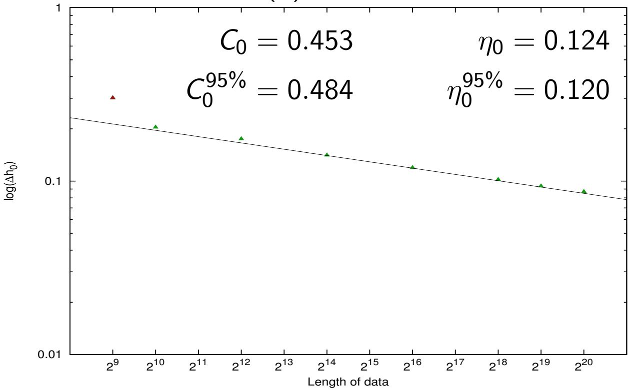

Figure: The spread of generalized Hurst exponent for fully correlated time series of various lengths

Multifractal effects in monofractal signals

joined dependence on L and γ

$$\Delta h(\gamma, L) = A(L)\gamma + B(L)$$

boundary conditions:

$$\Delta h(0, L) \equiv \Delta h_0 = 0.453 L^{-0.124}$$

$$\Delta h(1, L) \equiv \Delta h_{sh} = 0.678 L^{-0.114}$$

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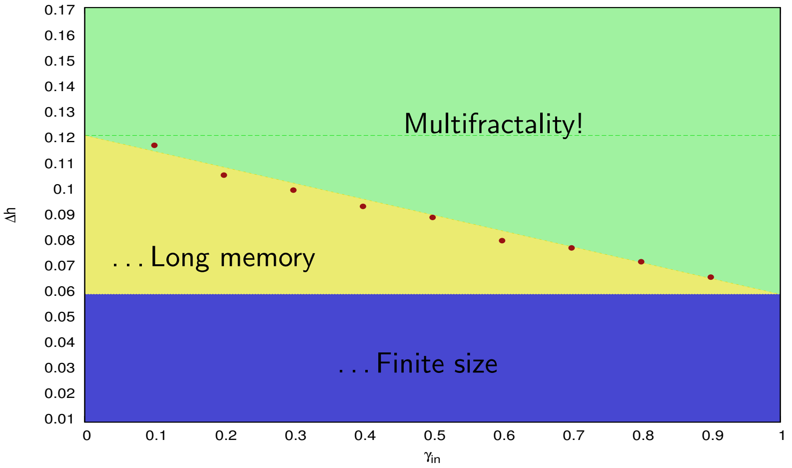
95% confidence level

$$\Delta h^{95\%}(\gamma, L) = A^{95\%}(L)\gamma + B^{95\%}(L)$$

boundary conditions:
$$\Delta h(0, L) \equiv \Delta h_0 = 0.484 L^{-0.120}$$
$$\Delta h(1, L) \equiv \Delta h_{sh} = 0.785 L^{-0.128}$$

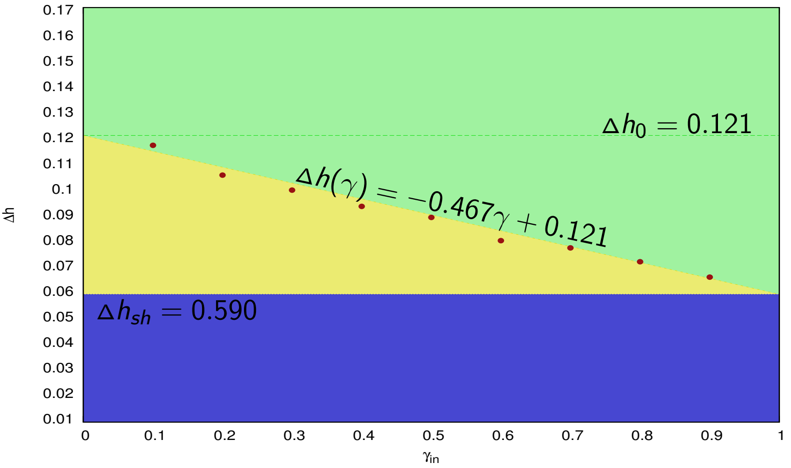
„Phase diagrams" for multifractality

Multifractal noise due to . . . effects



„Phase diagrams" for multifractality

Typical values for analytical expresion for $L = 1048576$



Analytical expression describing the $\Delta h(\gamma, L)$ dependence

$$\Delta h(\gamma, L) = C_{sh}^{95\%} L^{-\eta_{sh}^{95\%}} \gamma + C_0 L^{-\eta_0^{95\%}} (1-\gamma)$$

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$$\text{for } 0 \leq \gamma \leq 1$$

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$$\text{for } 0 \leq \gamma \leq 1$$

where

$$C_{sh} = 0.603 \qquad \eta_{sh} = 0.175$$

$$C_0 = 0.453 \qquad \eta_0 = 0.124$$

Analytical expression describing the $\Delta h(\gamma, L)$ dependence

$$\Delta h(\gamma, L) = C_{sh}^{95\%} L^{-\eta_{sh}^{95\%}} \gamma + C_0 L^{-\eta_0^{95\%}} (1 - \gamma)$$

$$\text{for } 0 \leq \gamma \leq 1$$

where

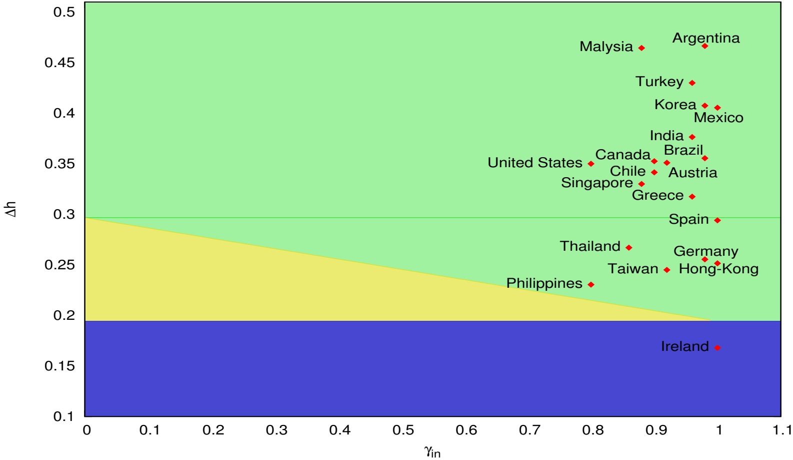
$$C_{sh} = 0.603 \qquad \eta_{sh} = 0.175$$
$$C_0 = 0.453 \qquad \eta_0 = 0.124$$

95% confidence level

$$C_{sh}^{95\%} = 0.631 \qquad \eta_{sh}^{95\%} = 0.171$$
$$C_0^{95\%} = 0.484 \qquad \eta_0^{95\%} = 0.120$$

„Phase diagrams" for multifractality

Analytical expression of the $\Delta h(\gamma, L)$ for $L = 2000$



- data taken from [L. Zunino, et.al. Phys.A **387** (2008) 6558-6566]

Conclusions

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- one should subtract MBN in any multifractal analysis to deal with *'true multifractality'* (differences in power-law scaling properties not depending on the data length and the memory magnitude (γ exponent) in TS)