The Action Principle in Market Mechanics

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Abstract

This paper explores the possibility that asset prices, especially those traded in large volume on public exchanges, might comply with specific physical laws of motion and probability. The paper first examines the basic dynamics of asset price displacement and finds one can model this dynamic as a harmonic oscillator at local "slices" of elapsed time. Based on this finding, the paper theorizes that price displacements are constrained, meaning they have extreme values beyond which they cannot go when measured over a large number of sequential periods. By assuming price displacements are also subject to the principle of stationary action, the paper explores a method for measuring specific probabilities of future price displacements based on prior historical data. Testing this theory with two prevalent stock indices suggests it can make accurate forecasts as to constraints on extreme price movements during market "crashes" and probabilities of specific price displacements at other times.

1. Introduction

Thank you very much for inviting me to present this paper at the 13th Econophysics Colloquium. It is truly an honor to be here at the University of Warsaw speaking with you. My goal today is to share my thoughts on the stock market as a physical system.

To quote Nash when he introduced his solution to the Riemannian manifold imbedding problem at the University of Chicago in 1955, "I did this because of a bet."[1] About 18 months ago I asked a colleague who is not only a legal scholar but also an economist what he thought about the application of Hamilton's principal function and principle of stationary action to assessing the structure of the stock market.[2] His response, if I remember correctly, went something like this: "I think this is silly." He went on to paraphrase von Mises, reminding me that economics is not physics.[3] But he also admitted that his macroeconomics professor had once said, "All economics today is just some form of 19th century physics. It's really going to get interesting once they start applying *20th* century physics!"[4] I took this as a

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[1] Sylvia Nasser, A Beautiful Mind: The Life of Mathematical Genius and Nobel Laureate John Nash 157 (2001). For the proof, see John Nash, *The Imbedding Problem for Riemannian Manifolds*, 63 Annals of Math. 20 (1956).

[2] *See* William R. Hamilton, *First Essay on On a General Method in Dynamics*, Phil. Trans. Royal Soc'y 95 (1835); William R. Hamilton, *Second Essay on On a General Method in Dynamics*, Phil. Trans. Royal Soc'y 247 (1835).

[3] *See, e.g.*, Ludwig von Mises, *Social Science and Natural Science*, 7 J. Soc. Phil. & Jur. 240, 245 (1942).

[4] *Cf.* Sitabhra Sinha, *Why Econophysics?*, in Econophysics & Economics of Games, Social Choices and Quantitative Techniques 157 (Banasri Basu et al. eds., 2010) ("[T]he pioneers of neoclassical economics had borrowed almost term by term the theoretical framework of classical physics in the 1870s to build the foundation of their discipline."). For a thorough history of the neoclassical adoption of classical physics developed primarily in the 19th century, see Philip Mirowski, *Physics and the*

challenge, but I also found inspiration in it. So this was, perhaps, more of a self-imposed dare than a bet, but what I have to present to you today was inspired by this short exchange.

I am neither a physicist, an economist, nor a mathematician. I am an American lawyer who is currently in an academic position although I formerly held certain leadership positions with the U.S. government relating to tax policy and administration. I originally asked that question of my colleague because at the time I imagined a model that explained how markets move, especially at extremely local points in time. It took me over 18 months to teach myself the appropriate mathematics and physics and I'm hoping I've learned at least enough to begin expressing this idea as a formal theory.

The principal project of econophysics, as I understand it, is to apply knowledge and techniques typically found in physics to the disciplines of finance and economics. Thus, the project implicitly assumes that finance and economics, to some extent, comply with (even if they are not necessarily subject to) physical laws. The theory expressed here tends to follow Dirac and Feynman in many respects. Yet, there are certain modifications necessary to ensure that the theoretical results match historical data; results that would not yield from a direct mapping of theoretical work on quantum mechanics to this work on market mechanics. I will highlight those modifications throughout.

The theory assumes three key postulates:

I. Observed market mechanics are the result of local systems that can be defined by their respective changes in price and time coordinates.

II. The only force directly affecting an asset's price displacement is the net force created by the trading activities of buyers and sellers that is transported by the superposition of respective wave functions at each instance of elapsed time.

III. The asset's action is the square of the phase displacement of the wave function.

'Marginalist Revolution,' 8 CAMBRIDGE J. ECON. 361 (1984).

From these postulates, I attempt to follow logical conclusions to see if the results match historical price data. As I will share in the final section, the results from this assumption tend to match historical data—at least for the few assets against which I have tested the theory. If others find this theory worth testing with other assets and other periods, and such tests support the theory, it is possible this work might serve as a seed that furthers the principal project of econophysics as a school of thought.

It's my hope that this theory also inspires a better understanding of market mechanics, especially asset and options pricing theory. While I do not explore these areas here it seems only natural that other work specifically examining asset and options pricing might benefit from this different point of view.

I'll begin in § 2 by exploring the basic dynamics of an asset's price displacement at each instance in time. In § 3, I'll investigate the forces that drive price displacement. In § 4, I'll explore the consequences of the third postulate, that the square of the phase displacement of the net wave function relating to price displacement is equivalent to the action of the asset. In § 5, I'll derive probability measures from the dynamics discussed. In § 6, I'll attempt to test the theory by comparing results obtained from traditional analyses with those obtained solely from the theoretical model. Initial comparisons with historical asset price data appear promising. I will close in § 7 with a brief summary and recommendations for future research.

2. POSTULATE I: LOCAL SYSTEMS

A. Elapsed Time and Price Displacement

Let's first explore the basic dynamics of an asset's change in price, i.e., its price displacement. Displacement generally involves moving something from one position to another over some elapsed period. Therefore, to arrive at the most basic dynamic of an asset's price displacement we must first try to understand the most basic instance of elapsed time.

Assume some elapsed period t, where t denotes a change in time from t_A to t_B, or $t := \Delta t = t_B - t_A$. If we divide this period by some integer n we get n-number of time slices, each of which we'll call τ. If n is very large then each time slice is very small.[5]

We can identify each time slice according to an index ϵ that runs from the states A to B such that

$$\epsilon \in \{A = 1 < 2 < \cdots < n-1 < n = B\}.$$

The identifier τ_ϵ then indicates the time slice that begins with the state ϵ; that is, $\tau_A = \tau_1 = t_1 \to t_2$, $\tau_2 = t_2 \to t_3$, $\tau_3 = t_3 \to t_4$, etc., until we finally get to $\tau_{n-1} = t_{n-1} \to t_n = t_B$. Also, since $\tau \neq dt$, the latter being the limit as t approaches zero, each time slice no matter how small has a beginning and an end time with some relevant meaning. Therefore, any τ-related derivatives of position will end up being averages from τ_ϵ to $\tau_{\epsilon+1}$.

The question then becomes, "What is the most basic displacement dynamic occurring at each time slice τ?" From an observation of assets traded on public exchanges it appears that the most basic displacement dynamic at each time slice is that the asset is either moving up or down in price to varying degrees.[6] Let's call an "up" movement positive and a "down" movement negative. These are arbitrary but intuitive directional assignments. As an asset moves in this way it travels not through traditional *space* but through the dimension of *price*. Let's call this price displacement x defined as the change in price from the asset's starting price x_A to its ending price x_B. Therefore, $x := \Delta x = x_B - x_A$.

B. *Changes in Price and Time*

If we imagine a coordinate system with t as the abscissa and x as the ordinate we get a standard-looking stock chart with time running from left to right and the price running up and down. This system produces the generalized coordinates (t_ϵ, x_ϵ). A consequence of this coordinate representation is that price displacements can be thought of as functions of elapsed time, or $x(t)$.

Elapsed time and price displacements are homogeneous even though objective time and prices are not. There is a significant difference between \$100 and \$50. The two are distinguishable in relevant ways and are, therefore, inhomogeneous. Yet, there is no difference between the displacements $(x_{A_1} = \$90, x_{B_1} = \$100)$ and $(x_{A_2} = \$40, x_{B_2} = \$50)$. In both instances the price displacements are $x = x_B - x_A = \$10$. In all relevant ways the two displacements are indistinguishable and, therefore, homogeneous.

The same is true of elapsed time. The five-day period between 14 March and 18 March is chronologically identical to the five-day period between 25 November and 29 November.

Therefore, the *changes* in price and time are invariant under translation, even though prices and time themselves are not. If the changes in price and time are homogeneous then they are also isotropic, meaning they are invariant under rotation (i.e., we can rotate the axes so as to swap time and price as the abscissa and ordinate and changes in price still remain functions of elapsed time). For these reasons, we can always assume both t_A and x_A are zero in relation to t_B and x_B, respectively, even if we rotate our coordinate system. The only difference in the coordinates is that t will always be

[5] These slices need not be precisely equal intervals, but for our purposes we will assume they are. *Cf.* R. P. Feynman, *Space-Time Approach to Non-Relativistic Quantum Mechanics*, 20 Rev. Mod. Phys. 367, 382 (1948) ("Clearly, any subdivision into instants [$t\epsilon$] will be satisfactory; the limits are to be taken as the largest spacing, [$t_{\epsilon+1} - t_\epsilon$] approaches zero.").

[6] For this reason, price displacement and all functions of it are vector quantities even though we use scalar notation throughout, e.g., $x \equiv \vec{x}$ and $f(x) \equiv \vec{f}(x)$.

positive since time only moves in one direction for our purposes here, while x might be positive, negative, or zero since prices move up, down, or experience no change.

3. Postulate II: External Forces

A. Forces and Their Balance

We know from experience that buyers and sellers interact with an asset and each other at each time slice τ.[7] Let's call a buyer's activity α since the activity is physically manifested in what's called an "ask" and a seller's activity β since the activity is physically manifested in what's called a "bid." Because we've decided to make the up direction positive and the down direction negative, we can notate a buyer's activity as $+\alpha$ and a seller's activity as $-\beta$.

We assume that the price displacement for some period t is proportional to the sum of buyers' and sellers' activities, or

$$x \propto [\alpha + (-\beta)] = \alpha - \beta,$$

since these are the only activities that affect the asset's movement through the price dimension. Consequently, if $\alpha = \beta$ then the actions cancel, producing a price displacement $x = 0$ for that period. However, if $\alpha > \beta$ then $x > 0$ (positive), and if $\alpha < \beta$ then $x < 0$ (negative).

Based on this we can define a variable p as the *relative frequency* of an asset experiencing net "up" activity ($\alpha > \beta$), and its complement q as the relative frequency of experiencing net "down" activity ($\alpha < \beta$), for an appropriate-sized sample of elapsed periods t.

More specifically, we can think of the dynamic $\alpha - \beta$ as representing some *net force* generated by the interactions of buyers and sellers.[8] From this we can assume that the net force is proportional to the change in price,[9] or $F \propto x$. Here, F is the net force resulting from the superposition of F_α and F_β, or

$$F = F_\alpha + [-F_\beta] = F_\alpha - F_\beta.$$

We can also assume that something "carries" these forces to produce the observed price displacement of an asset. For now, we will refer to this net "carrying thing" as ψ, where ψ is the result of the superposition of the "carrying thing" of the buyers' force and that of the sellers'.

Let's return to our proportional assumption that $F \propto x$. Since both F and x are time functions, we know per Newton's second law that $F = m\ddot{x}$, where m can be thought of as an inertial coefficient unique to each asset and $\ddot{x}$ is the second time derivative of the price displacement. This leaves us with $m\ddot{x} \propto x$, which suggests that in the form of an equality the right-hand side is multiplied by a coefficient we'll call k that is somehow related to the inertial coefficient m, although its not necessary for us to figure out that exact relation right now.

We would like an expression that balances the force on the left-hand side of the equation with the force on the right-hand side. Such balance allows any energy elements we wish to discuss to be independent of the "path" the asset takes through the relative configuration space between states A and B since the number of possible paths are infinite.

Additionally, since we are concerned only with the *changes* in elements such as price, time, and force, we essentially "reset" our system every period t. The net force represented by

[7] *See also* Rosario N. Mantegna & H. Eugene Stanley, An Introduction to Econophysics: Correlations and Complexity in Finance 8 (2000) ("Financial markets are systems in which a large number of traders interact with one another and react to external information in order to determine the best price for a given item.").

[8] It is critical to remember that the *interactions* of buyers and sellers move an asset through price, not the buyers and sellers themselves. For this reason, the individual incentives of buyers and sellers are immaterial to this theory.

[9] Recall that x is always a function of t. Since the net force can be thought of as a function of the price displacement, it is also a function of elapsed time.

the left-hand side of the equation displaces the asset through the price dimension a distance of exactly x in a particular direction starting from $x_A = 0$ to x_B. For the next period t to begin, the asset must first return to its zero starting point, once again traversing the exact same distance x only this time in the opposite direction from x_B back to $x_A = 0$. Therefore, if the left-hand side represents a force in one direction we must have the right-hand side be in the opposite direction.

Accordingly, the statement representing the balance of forces for any price displacement x over an elapsed period t becomes

$$m\ddot{x} = -kx. \tag{1}$$

This is a very familiar equation since it is one describing the motion of a harmonic oscillator.[10] An oscillator only goes in two directions: up or down. So this description is at least consistent with our observation that at each time slice τ the basic dynamic is that the asset goes up or down in price. That's very fortunate.

B. *Solutions for the Price Displacement*

It's also fortunate because we already know a solution to Eq. (1). The complex solution is

$$\psi_B = \mathcal{R}(\cos\theta + i\sin\theta),$$

where θ is the principal value of the argument of the complex number ψ at t_B, or $\arg(\psi_B) = \theta$.

Recall that we are dealing here with changes in elements in specific states, i.e., the differences between a phenomenon at state A and at state B. We should, therefore, use the same lens in examining all related phenomena discussed here. The complex number ψ becomes $\psi := \Delta\psi = \psi_B - \psi_A$. Because we deal only with the net differences, we can assume that $\psi_A = i\mathcal{R}$ since x_A and t_A are both zero at state A, and $\psi_B = \mathcal{R}(\cos\theta + i\sin\theta)$ at state B.

We must examine the principal value of $\arg(\psi)$ in the same way, so

$$\arg(\psi) := \Delta\arg(\psi) = \arg(\psi_B) - \arg(\psi_A).$$

In other words, the *net* principal value of $\arg(\psi)$ is the difference between the principal value of $\arg(\psi)$ at t_A and the principal value of $\arg(\psi)$ at t_B.

The principal value of the argument of ψ is defined geometrically as the angle in the complex plane from the positive $\Re$-axis (our price axis here) to the vector representing ψ_B. For us, this vector has the scalar length $\mathcal{R}$, which is the modulus of ψ_B. Defined this way, at $t_A = 0$ where x_A is always considered zero, the principal value of $\arg(\psi_A) = \theta_A = \frac{\pi}{2}$. This means the principal value of $\arg(\psi_B) = \theta_B$. Denoting $\phi := \Delta\theta = \arg(\psi_B) - \arg(\psi_A) = \theta_B - \theta_A$ we get

$$\phi = \theta - \frac{\pi}{2}.$$

Thus, we see that the principle value of the phase displacement ϕ is always the complement of the principle value of the phase θ.

This means the real solution for Eq. (1) (the price displacement we actually observe) is

$$x_B = x = \mathcal{R}\cos\theta = \mathcal{R}\sin\phi. \tag{2}$$

For our purposes, we can express this simply as $x = \mathcal{R}\sin\phi$. Thus, the complex solution in terms of ϕ becomes

$$\psi_B = \mathcal{R}(\sin\phi + i\cos\phi). \tag{3}$$

Set up this way, ψ is the complex superposition of things that carry the forces of buyers and sellers with a resulting amplitude of $\pm\mathcal{R}$. This is consistent with our assumption that some "carrying thing" is responsible for trans-

[10] More generally, the system can be expressed as a Van der Pol equation $\ddot{x} - \mu(1 - x^2)\dot{x} = -x$ with the constant μ at or very near zero keeping the limit cycle close to circular since, as we shall see in § 4, the theory relies on either the total or very near total conservation of energy with any net energy introduced into the system wholly converted to motion in the price dimension.

porting the net force generated by the trading interactions of buyers and sellers that ultimately effects a price change for an asset. As with many things in the physical world, the thing that carries the force in our theory is akin to a wave, or at least its analog.[11] Therefore, it is appropriate to discuss the principal value of the argument of the complex number ψ as the "phase" and its complement as the "phase complement." Yet, keep in mind that the phase complement really represents the *change* in the phase from t_A to t_B. For this reason, we refer to the phase complement ϕ as the "phase displacement."

Throughout this paper we are ultimately interested in the price displacement x, which we know is equivalent to x_B given that x_A can always be considered zero. Yet, ψ_A cannot be considered zero since $\psi_A = i\mathcal{R}$, so $\psi \neq \psi_B$. Still, we will use ψ to denote ψ_B for the remainder of this paper with the clear understanding that when we use ψ hereafter we are really talking about the complex number ψ at state B and not the change in the complex numbers from states A to B.

Looking at our solution for the price displacement in Eq. (2) we can make two related observations. The first is that there exists a unique phase displacement measure ϕ for each possible price displacement over period t. The second is that the net movement in price over period t is some maximum absolute measure multiplied by a function that oscillates between -1 and $+1$. In other words, $x_{\min} = -\mathcal{R}$ and $x_{\max} = +\mathcal{R}$. This has significant implications for our theory of market mechanics as it suggests there is some net price displacement measure beyond which an asset cannot go for a defined period. The absolute price displacement can be greater than $\mathcal{R}$ for a period less than t (e.g., $t_{(n-j)} - t_A$ where j is a positive, non-zero integer); however, it cannot be greater than $\mathcal{R}$ for $t_B - t_A$. This means an asset's net price movement is *constrained* for a specified period; not by some external regulation, but by the properties of the asset itself.

Returning to our relative frequency variable p we might want to ask at this point, "What are other ways we can describe the relative frequency of something being positive or negative?" As we've just stated, $\sin\phi$ oscillates between -1 and $+1$ in the range $\phi = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Therefore, we want a function in terms of p that does the same.

We know that a relative frequency, by definition, has a range on the closed interval $[0, 1]$. Recall that we've defined p as the relative frequency of the price displacement being in the positive direction. This means the relative frequency of the price displacement being in the negative direction is $q = (1 - p)$. If $p = 1.00$ then $\phi = \frac{\pi}{2}$. If $p = 0.75$ then $\phi = \frac{\pi}{4}$. If $p = 0.50$ then $\phi = 0$ (forces are balanced). If $p = 0.25$ then $\phi = -\frac{\pi}{4}$. Finally, if $p = 0.00$ then $\phi = -\frac{\pi}{2}$.

This means

$$\phi = \pi\left(p - \frac{1}{2}\right).$$

Substituting this value into our solution for the price displacement in Eq. (2) yields

$$x = \mathcal{R}\cos(q\pi) = -\mathcal{R}\cos(p\pi). \qquad (4)$$

If we know the average value $\bar{p}$ for the most local period τ and the average absolute price displacement $|\bar{x}|$ for the defined period t, then given an appropriate set of sequential periods t we should be able to approximate the absolute constraint of the asset's price displacement for any average period with the equation

$$|\mathcal{R}| = \left|-\frac{|\bar{x}|}{\cos(p\pi)}\right| = \left|-|\bar{x}|\sec(p\pi)\right|. \qquad (5)$$

Again, this will only give us an approximation of $|\mathcal{R}|$. We want an average of p for all τ since

[11] Although it is more accurate to refer to these as *wave-like functions*, for the sake of brevity we will call them *wave functions* throughout.

we are trying to approximate the relative frequency of the asset generally, and we want an average for $|x|$ for only our defined t since that is the specific period for which we seek the absolute price constraint $|\mathcal{R}|$. We will return to this expression and calculate approximations of the absolute price constraint for sample assets in the final section of this paper.

C. *From Complex Solutions to the Real Solution*

The real solution for the price displacement in Eq. (2) is a result of our solving Eq. (1) and ignoring the imaginary parts of the complex solution ψ in Eq. (3). This approach implies that we only observe one part of a much richer physical reality, the imaginary part of which somehow still lingers in existence but beyond our ability to observe it. Such an approach certainly stirs the imagination to conceive of a "hidden dimension" that exists beyond our senses yet is still responsible for the phenomena we experience in our daily lives.

As fascinating as the implications of this approach might be, there is an explanation besides that of a lingering hidden dimension that is just as mathematically legitimate; one that regards the imaginary elements of the complex plane as a mere mathematical tool to help us understand the reality we see, but not as a physical reality itself. Let's examine an approach that yields the same result as Eq. (2) for an asset's price displacement without admitting a lingering hidden dimension in our physical reality.

So far we have only discussed ψ as the solution for Eq. (1). Mathematically, there also exists a solution for Eq. (1) that is the mirror image of ψ reflected about the $\Re$-axis:

$$\psi^* = \mathcal{R}(\sin\phi - i\cos\phi).$$

This is the complex conjugate of ψ.

Since ψ^* is a mirror reflection of ψ, we can assume both have the same phase displacement measure of ϕ. Therefore, we can consider the price displacement x to be the one-dimensional median between ψ and ψ^* in the complex plane. Because the distribution of planar area is symmetric between ψ and ψ^*, we can express this median as

$$x = \frac{1}{2}(\psi + \psi^*). \tag{6}$$

We can say the same for x in the negative direction using $-\psi$ and $-\psi^*$.

Given our definitions of ψ and ψ^* we see that

$$\psi + \psi^* = 2\mathcal{R}\sin\phi.$$

Note that the imaginary terms $i\cos\phi$ in ψ and $-i\cos\phi$ in ψ^* cancel when added. Substituting this sum into Eq. (6) we recover Eq. (2).

By thinking of the price displacement as the median of the regions between the wave function and its complex conjugate, we arrive at the same solution using a mathematical contrivance that cancels the imaginary terms instead of admitting a lingering dimension that physically exists but which we simply ignore.

4. Postulate III: Action Principle

So far, we haven't added anything all that revolutionary to the discussion. Asset prices move up or down at each time slice. The force that moves an asset through the price dimension is the superposition of the forces from buyers and sellers interacting with each other through trading activities, and these forces are carried by something we mathematically describe as a complex wave function. Interesting and even intuitive observations perhaps, but not very helpful in describing market mechanics with specificity.

But here is where we introduce the "leap of faith;" one that is admittedly a Samuelsonian nightmare.[12] Our third postulate is that

[12] Paul A. Samuelson, "Maximum Principles in Analyti-

the square of the phase displacement ϕ^2 is the asset's action $\mathcal{S}$; the latter being a functional defined as the time integral of the asset's Lagrangian $\mathcal{L}$. In other words,

$$\phi^2 = \mathcal{S} = \int_{t_A}^{t_B} \mathcal{L}\, dt. \tag{7}$$

This is a modified adoption of Dirac and Feynman.[13] Both postulated that the phase is the action along a particle's path in space-time for quantum mechanics, or $\theta = \mathcal{S}$.[14] The modification made here is that the *square* of the phase *displacement* is the action along an *asset's* path in *price*-time dimensions for *market* mechanics, or $\phi^2 = \mathcal{S}$. We offer a justification for this postulate in the following subsection.[15]

cal Economics," in *Nobel Lectures, Economics 1969-1980*, p. 68, Editor Assar Lindbeck, World Scientific Publishing Co., Singapore, 1992.

> There is really nothing more pathetic than to have an economist or a retired engineer try to force analogies between the concepts of physics and the concepts of economics. How many dreary papers have I had to referee in which the author is looking for something that corresponds to entropy or to one or another form of energy.

[13] *See* P. A. M. Dirac, *The Lagrangian in Quantum Mechanics*, 3 Physik. Zeitschrift der Sowjetunion 64, 68 (1933); Feynman (1948).

[14] Assuming $\hbar = 1$. *See* George Gamow, Mr. Tomkins in Paperback 67 (Canto ed. 1993):

> [A]ll bodies in nature are subject to quantum laws, but the so-called quantum constant [$\hbar$] which governs these phenomena is very, very small [...]. For these balls here, however, this constant is much larger—about unity—and you may easily see with your own eyes phenomena which science succeeded in discovering only by using very sensitive and sophisticated methods of observation.

[15] The price displacement is a result of the phase displacement ϕ belonging to the complex wave function ψ. In this way, we can express the phase displacement as a functional of the price displacement just as the action is a functional of the price displacement. Thus, we should express them formally as $\mathcal{S}[x(t)]$ and $\phi[x(t)]$, although we may abbreviate these as simply $\mathcal{S}$ and ϕ for concision if there is no risk of confusion.

A. *The Action and the Lagrangian Generally*

The Lagrangian is a way of describing a system as a function of the conditions at t_ϵ (or any initial time for a slice τ) relating to the price at the beginning of the time slice (x_ϵ) and the first derivative of that price ($\dot{x}_\epsilon$). It contains the complete information of both the system and the effects of forces acting upon the system.[16]

Alternatively, the Lagrangian is often defined as the difference between the kinetic energy K and the potential energy V of a system, each measured at t_ϵ, since K expresses energy in terms of $\dot{x}_\epsilon$ and V in terms of x_ϵ. Therefore, $\mathcal{L} = K_\epsilon - V_\epsilon$.[17]

This theory posits that an asset contains none of its own force and, therefore, none of its own energy. Any displacement of the asset in price is a result of work done on the asset, which is equal to the net external force over the amount of any displacement. The theory further holds that there is only one aggregate source that generates this net external force and introduces it to the asset: buyers' and sellers' trading interactions.

From the previous sections we see that at each time slice τ the asset moves linearly in price. This is consistent with the principle of stationary action. As applied here, the principle holds that for each elapsed time t the "path" taken by an asset (i.e., the curve traced out in the relative configuration space of price) between times t_A and t_B is the one for which the action does not change (i.e., is stationary) under small changes in the relative configuration of the asset related to the relative price dimension. This is usually expressed as $\delta\mathcal{S} = 0$.

In our dimension of price, such a path is

[16] Dirac would most likely consider the Lagrangian not as a function of the asset's initial coordinates and its first derivative, but instead as a function of the asset's price at time t_ϵ and its price at time $t_{\epsilon+1}$. *See* Dirac at 68.

[17] Note that the kinetic energy is simply the integral of the left-hand side of Eq. (1) and the potential energy is the negative integral of the right-hand side, both with respect to price displacement.

expressed as a straight line. Since the price dimension is our vertical coordinate axis, we can go further and state that such a path is a straight line in either the up or down direction, although under rotation we can generalize it as the positive or negative direction. This is consistent with the oscillating dynamic expressed earlier.

It is important to note this is only consistent with observed market mechanics when $t = \tau$; that is, the asset moves strictly up or down in price only at local time slices. What is "local?" Well, that depends on what measure we wish to give n, which will dictate our definition of τ. Let's make $n = 5$. If t_A is Monday morning and t_B is Friday afternoon then at $n = 5$ and $\tau = 1$ day, experience tells us that the asset does not move in a straight line directly from t_A on Monday to t_B on Friday, although a straight line from t_A on Monday to t_B on Friday would still be an example of stationary action for the price displacement between *only* those two points in time. Yet, if $n = 1$ and $\tau = 5$ days, then the asset does strictly move up or down in price during that period.

Thought of another way, the action—again, the time integral of the Lagrangian—over any region of our price-time coordinate system must be stationary for any small changes in the coordinates in that region. If we keep dividing the regions until we get to a collection of time slices, we observe this stationary action as the binary "up-down" oscillations of typical market mechanics at each time slice τ, which again is the elapsed time from t_ϵ to $t_{\epsilon+1}$. Therefore, at each point in our price-time coordinate system there exists a Lagrangian that is a function of its coordinates and their first derivatives with respect to time.

Does this mean we can't determine the action between t_A and t_B when the elapsed time is not local? No, in fact, quite the opposite. It simply means we should examine the general dynamics of market mechanics by first examining the specific dynamics at each slice τ between t_A and t_B and then sum over all periods τ to find measurable results that match observed data. In other words,

$$\int_{t_A}^{t_B} \mathcal{L}dt = \int_{t_0}^{t_1} \mathcal{L}dt + \int_{t_1}^{t_2} \mathcal{L}dt + \cdots + \int_{t_{n-1}}^{t_n} \mathcal{L}dt, \tag{8}$$

where $\tau_\epsilon = t_{\epsilon+1} - t_\epsilon$ just as on a larger scale we find $t = t_B - t_A$.

The Lagrangian over each time slice τ_ϵ in our price-time coordinate system makes some contribution to the total price displacement x measured between t_A and t_B.[18] Observe, however, that the price displacement is not only a result of the phase displacement of the wave function ψ. It is also a result of the phase displacement belonging to the complex conjugate of the wave function ψ^*, which we'll denote here as ϕ^* to avoid confusion.[19] In fact, both ϕ and ϕ^* contribute in equal measure to x since, as we showed in the previous section, a complex number and its complex conjugate share the same real value, which in this case is the price displacement. So if the Lagrangian at each point makes some contribution to the total price displacement x measured between t_A and t_B so does *twice* the phase displacement, or 2ϕ.

Look again at Eq. (8). Notice that the local elements that sum to the total time integral of the Lagrangian between t_A and t_B are each just a small contribution to the total action since the total action for the period between t_A and t_B is defined as the total time integral of the Lagrangian for that period. Hence, we can state that each element $\int_{t_\epsilon}^{t_{\epsilon+1}} \mathcal{L}\, dt = d\mathcal{S}$.

Additionally, there is a phase displacement for each slice τ_ϵ and its conjugate twin that contribute equally to the price displacement

[18] This is similar to Dirac's approach. Dirac at 69.

[19] Technically, the complex wave function ψ has the conjugate ψ^* and both have identical phase displacements ϕ; however, it's easier to talk about the phase displacements attributed to each by denoting them ϕ and ϕ^*.

over that time slice, or $2\phi[x(\tau_\epsilon)]$. Thus, we can state that there will always exists twice the phase displacement for the time slice τ_ϵ given any small change in the Lagrangian's contribution to that part of the action with respect to a small change in the phase displacement over the same period, or more formally

$$\frac{d\mathcal{S}}{d\phi} = 2\phi. \tag{9}$$

We've already established from Eq. (8) that $\int_{t_\epsilon}^{t_{\epsilon+1}} \mathcal{L}\, dt = d\mathcal{S}$. Therefore, we can make substitutions such that

$$\int_{t_k}^{t_{\epsilon+1}} \mathcal{L}\, dt = 2\phi\, d\phi.$$

Adding up (integrating) the left-hand side from t_A to t_B gives us the total action $\mathcal{S}$. Doing the same to the right-hand side gives us $2\left(\frac{\phi^2}{2}\right)$. We can express this simply as

$$\mathcal{S} = \phi^2. \tag{10}$$

This is the justification—although admittedly not a rigorous mathematical proof—for our third postulate expressed in Eq. (7). The stationary action attributable to the price displacement x is the square of the phase displacement of the wave function.

B. *Measuring the Action*

The next question becomes, "What is the measure of this action, and by extension, the measure of the phase displacement?" We attempt to answer this by again examining the dynamics at each time slice.

At each slice τ the net external force causes a tiny amount of work to be done with respect to a tiny displacement in price, especially when n is very large, or $F(\tau_\epsilon) = \frac{dW}{dx(\tau_\epsilon)}$. To figure out the total amount of work done on the asset over the period we take the price integral of the net external force, or $W = \frac{1}{2}Fx$.

The work W is equivalent to the total amount of energy necessary to be introduced into the asset from external forces in order to make the asset move in price. Since the asset has none of its own energy and does not store energy, any energy used to move the asset through the price dimension must come from the net energy introduced into the asset by the external interactions of buyers and sellers. In other words, the kinetic energy K resulting from any price displacement must come from an external potential V. This means the final amount of kinetic energy must equal the initial amount of potential energy as measured over any elapsed period.

Consequently, we can hold that the total work performed on the asset equals the total potential initially introduced into the asset. This is also equal to the final energy from the movement of the asset through the price dimension, which is the kinetic energy. In other words, $W = V_\epsilon = K_{\epsilon+1}$. For this to be true, $V_{\epsilon+1}$ and K_ϵ must equal zero, with V and K "trading off" energy over each elapsed period but the sum of V and K always equaling W for that time slice. Since $K_\epsilon = 0$, the Lagrangian becomes

$$\mathcal{L} = K_\epsilon - V_\epsilon = 0 - V_\epsilon = -V_\epsilon.$$

Another way of looking at this is to consider the Lagrangian of a harmonic oscillator, which is well known to be $\mathcal{L} = \frac{1}{2}(m\dot{x}^2 - kx^2)$. This is just another representation of the difference between the kinetic and potential energies since the kinetic energy for any system is $\frac{m\dot{x}^2}{2}$ and the general potential for a harmonic oscillator is $\frac{kx^2}{2}$. Rotating our coordinate system, which is allowable for any system with isotropic coordinates, we see that $\dot{x} = 0$ for all values of x_ϵ, yielding the Lagrangian $\mathcal{L} = -\frac{kx^2}{2} = -V_\epsilon$.

Put another way still, at $t = t_\epsilon = \tau_\epsilon = 0$ (no elapsed time) $\dot{x}(0) = 0$ and from Eq. (2) it follows that $x(0) = \mathcal{R}_\epsilon$. This last term is the maximum absolute displacement potential at the start of that particular slice, or $x(\tau_\epsilon) = \mathcal{R}_\epsilon$, even though $x(\tau_\epsilon) \leq \mathcal{R}_\epsilon$ given our standing assumption that $\tau_\epsilon > 0$. The Lagrangian then becomes

$$\mathcal{L} = \frac{1}{2}(m(0)^2 - k\mathcal{R}_\epsilon^2) = -\frac{k\mathcal{R}_\epsilon^2}{2} = -\frac{kx^2}{2},$$

which again gives us $-V_\epsilon$.

We've made a bit of progress. We now know that the Lagrangian is the negative initial potential energy generated by the buyers and sellers at any time slice τ_ϵ. But what is the measure of the potential energy? We know it must be the same as the average measure of work, which is equal to $\frac{1}{2}Fx$. We also know from Newton's second law that $F = m\ddot{x} = \frac{mx}{t^2}$ since $\ddot{x} = x/t^2$ when taken as an average. Therefore,

$$\frac{1}{2}Fx = \frac{mx}{2t^2}x = \frac{mx^2}{2t^2}.$$

But this is the same as the measure of the average kinetic energy of a system since the average $\dot{x}$ is always x/t. It seems, therefore, that $K_{\max} = V_{\max}$. Since $K_{\max}$ implies that $V_{\min} = 0$, $V_{\max}$ implies that $K_{\min} = 0$, and $V_\epsilon \equiv V_{\max}$, we can conclude that

$$V_\epsilon = \frac{kx^2}{2} = \frac{mx^2}{2t^2}$$

as an average measure for $t = \tau_\epsilon$. Consequently, we can express the Lagrangian of an asset as

$$\mathcal{L} = -V_\epsilon = -\frac{mx^2}{2t^2}.$$

Because the action is defined as the time integral of the Lagrangian, we can express the action as

$$\mathcal{S} = \int_{t_A}^{t_B} -\frac{mx^2}{2t^2}\, dt = \frac{mx^2}{2t}. \tag{11}$$

C. *Deriving the Inertial Coefficient*

Given our postulate in Eq. (7) and our measure of the action in Eq. (11), we can now express the phase displacement as the square root of the action, or

$$|\phi| = |x|\sqrt{\frac{m}{2t}}.$$

We've already shown that for every price displacement there exist two phase displacements, each of equal absolute value. In other words, $|x| \implies 2|\phi|$.

The maximum absolute price displacement $|\mathcal{R}|$ occurs when $|\phi| = |\frac{\pi}{2}|$. This single absolute price displacement implies twice the phase displacement at $|\mathcal{R}|$. This gives us $2|\phi| = 2\left|\frac{\pi}{2}\right|$. Thus, we have the proposition

$$|\pi| = |\mathcal{R}|\sqrt{\frac{m}{2t}}.$$

This yields

$$m = 2t\left(\frac{|\pi|}{|\mathcal{R}|}\right)^2. \tag{12}$$

To be clear, m is a statistical result. However, if our sample of the number of periods t is the appropriate size the sample should begin to approach the "population" for that period t. As this happens, the statistical value of m for period t should approach the actual value for the asset.[20]

The question then becomes, "What is the proper population for an asset given its dynamic state?" For example, the U.S. S&P 500 Index has data going back decades. If one were to take data from, say, 1980 through 2016 and call that a "complete population," the very low values of the early data would excessively skew the results so as to leave calculations for the more current data without meaning. Therefore, if this theory is to have any practical signifi-

[20] From the equations it is obvious that the inertial coefficient m scales with the period t, but probably not linearly.

cance we need to ensure for each market that we sample data far enough back to be statistically relevant, but not so far back that the earlier data weigh down predictions for the asset in its current price state. I leave the task of defining the "current price state" and the appropriate population of each asset to others as such experimental explorations are beyond the theoretical scope of this paper.[21]

5. Deriving Probabilities

Next, let's investigate how we might derive certain probability-related measures, specifically normalized Gaussian distributions and normalized error functions, given the information we have so far.

A. Normalized Gaussian Distributions

Let's first examine how we can transform the complex price displacement solution into a probability distribution. Let's start by trying to answer the question, "What is that probability of the complex number ψ occurring?" Further, given our third postulate, we'll want to answer this question as a function of ϕ. Recall the complex solution to Eq. (1) as a function of the phase displacement from Eq. (3): $\psi(\phi) = \mathcal{R}(i\cos\phi + \sin\phi)$. Using Euler's formula, Eq. (3) becomes

$$\psi = \mathcal{R}\ ie^{-i\phi}.$$

We see that (1) ψ is the unique complex wave function for period t, (2) $ie^{-i\phi}$ is the wave form, and (3) $\mathcal{R}$ is the amplitude, or maximum absolute displacement, of the wave form.

This becomes the following probability measure with only minor modifications:

$$\Pr(\phi) = Q\ ie^{(-i\phi)^2} = Q\ ie^{-\phi^2}.$$

Here, (1) $\Pr(\phi)$ is the probability function of the phase displacement, (2) $ie^{-\phi^2}$ is the Gaussian form (a modified wave form), and (3) Q is the maximum height of the Gaussian form, which we will also refer to as a "normalizer," where $0 \leq Q \leq 1$. The unit i tells us that this probability distribution, at least for now, exists only on the complex plane along the $\Im$-axis.

Let's first take a look at the Gaussian form $ie^{-\phi^2}$. Alone, it has a maximum height of $\Pr(\psi) = i$, which we can interpret for now as unity, at $\phi = 0$. In other words, the form as expressed states that there is a 100 percent chance that the phase displacement is zero. We know from experience that this is not the case (i.e., it's not zero all the time). This means the Gaussian form must be *normalized*, or divided by the sum of all possible values of ϕ. This sum can be expressed by the integral

$$\int_{-\infty}^{+\infty} ie^{-\phi^2} d\phi = i\sqrt{\pi}.$$

As we must divide the imaginary Gaussian $ie^{-\phi^2}$ by the result $i\sqrt{\pi}$, the imaginary units cancel leaving a normalized probability measure along the $\Re$-axis instead of the imaginary. As a result we can define the maximum height of the generically normalized Gaussian as $Q = \frac{1}{\sqrt{\pi}}$ and our initial probability of finding ψ becomes the statement

$$\Pr(\psi) = \frac{e^{-\phi^2}}{\sqrt{\pi}}. \tag{13}$$

Note that this expression is a *normal probability density function* with standard deviation $\sigma = \frac{1}{\sqrt{2}}$. This is more accurately the probability that one would find a specific value of the wave function equal to a value chosen arbitrarily. If we call this arbitrary value Ψ, we can express this as

$$\Pr'(\psi = \Psi) = \frac{e^{-\phi^2}}{\sqrt{\pi}}. \tag{14}$$

Note also that the normalizer Q is only $\frac{1}{\sqrt{\pi}}$ if we seek the probability of ψ generically. If,

[21] In America, we call this "punting the football."

instead, we seek the probability of one specific component of the function—let's call that component c_1—then the normalizer will be the product of $\frac{1}{\sqrt{\pi}}$ and some function $f(\cdot)$ of the remaining components of ϕ. Let's call these remaining components $c_2, c_3, \cdots, c_n$. In other words,

$$Q = f(c_2, \cdots, c_n)\frac{1}{\sqrt{\pi}} \tag{15}$$

where $f(0) = 1$.

We see this if we replace ϕ^2 with $\mathcal{S}$ per our third postulate and want to know the probability of $|x|$ being some random non-negative value X.[22] This then becomes

$$\sqrt{\frac{m}{2\pi t}}\, e^{-\frac{mX^2}{2t}}. \tag{16}$$

Notice that when $X = 0$ the probability becomes the normalizer Q. This makes sense since the value of Q is the maximum point on the normalized Gaussian curve and it exists at $X = 0$ (no price displacement).

Yet, unlike with ψ, this is not a complete expression of the probability of the absolute price displacement being some arbitrary value X. Recall that up until now we were looking for the probability of finding a unique ψ as a function of ϕ. In all of the complex plane, there is one and only one propositional function to describe a unique ψ and that is found in Eq. (3). But we are now looking for the unique absolute price displacement $|x|$. If we think back to our complex plane we find that there is *not* one and only one propositional function to describe a unique $|x|$. There are two.

One is ψ, but the other is the complex conjugate ψ^*. Since $|x|$ results from the phase displacement associated with ψ and the phase displacement associated with ψ^*, we must add the ϕ-terms for both sides. Recall that we modified the wave function to arrive at the probability function. Thus, we can state that the wave function always implies a probability function if properly modified, or $\Pr(\psi) = Q\, e^{(i\phi)^2}$. Now we must also consider ψ^* so we must also examine the modified wave function implied by the complex conjugate to arrive at yet another probability function, or $\Pr(\psi^*) = Q\, e^{(-i\phi)^2}$.

Combining the complex wave function and its conjugate gives us $\psi \cdot \psi^* = |\psi|^2$. This doubles the phase displacement ϕ. Continuing our modified analogy we arrive at the implied probability function $\left(Q\, e^{(i\phi)^2}\right)\left(Q\, e^{(-i\phi)^2}\right)$. Since $e^{(i\phi)^2} = e^{(-i\phi)^2} = e^{-\phi^2}$, this is equivalent to $\left(Q\, e^{-\phi^2}\right)^2$. Therefore, it follows that

$$\Pr(|x|) = \left(Q\, e^{-\phi^2}\right)^2, \tag{17}$$

remembering, of course, that ϕ is a functional of the price displacement function x.

Alternatively, we can think of this as a more basic probability problem where we ask the question, "What is the probability of ψ and ψ^*?" Formally, this becomes $\Pr(\psi \cap \psi^*)$. We know how to solve this since it is proven that

$$\Pr(\psi \cap \psi^*) = \Pr(\psi) \cdot \Pr(\psi^*)$$

if ψ and ψ^* are independent events, which they are since they are unique and separate from each other. Because this is the same as the probability of $|x|$ since the phase displacements are functionals of the the price displacement function, it follows that

$$\Pr(|x|) = \Pr(\psi) \cdot \Pr(\psi^*).$$

This again gives us Eq. (17), since $\Pr(\psi) = Q\, e^{(i\phi)^2}$ and $\Pr(\psi^*) = Q\, e^{(-i\phi)^2}$.

Note that we are simply adding the two phase displacements together. Since the complements are in an exponential form we add by multiplying the two exponential functions, which in this case is equivalent to squaring the original exponential function. This is because the product of a series of exponential functions

[22] X must always be non-negative since it is an arbitrary value of the absolute price displacement function $|x|$.

is equal to the exponential function of the sum of a series of exponents, or

$$\prod_j \exp\left(a_j\right) = \exp\left(\sum_j a_j\right).$$

Given our third postulate, the exponential function thus becomes $(e^{-\phi^2})^2 = e^{-2\phi^2} = e^{-2\mathcal{S}} = e^{-2\left(\frac{mx^2}{2t}\right)} = e^{-\frac{mx^2}{t}}$. Since $Q = \sqrt{\frac{m}{2\pi t}}$, the probability of $|x|$ being some arbitrary value X becomes

$$\Pr(|x| = X) = \frac{m}{2\pi t}\, e^{-\frac{mX^2}{t}}. \tag{18}$$

This was similarly shown by Feynman in relation to quantum mechanics, but with a slightly different approach.[23] Both Feynman's approach and that taken here imply that the total probability measure of the displacement x for any region of relative configuration space bound by coordinates $(t_A, \Re_A, \pm\Im_A)$ and $(t_B, \Re_B, \pm\Im_B)$, where $\Re_\epsilon \equiv x_\epsilon$, is the product of two identically split regions of three-dimensional relative configuration space since the probabilities as we've constructed them are exponential functions relating to the phase displacement ϕ. The entire probability measure is the combination of two evenly-split regions with respect to ψ and ψ^* along the $\Re$-axis. We obtain this combination by multiplication.

In short, we must square the traditional probability function effectively doubling the action since there are two possible ways in the complex plane we can get the exact same absolute price displacement $|x|$. This approach does not violate the principle of unitarity since the maximum probability remains unity and the minimum zero.

B. *Normalized Error Functions*

So far we've discussed a method to find the probability that $\psi = \Psi$, but what if we seek the probability that $\psi < \Psi$? To find this we must add up all the possible values of the phase displacement from zero to the value of ϕ associated with the arbitrary complex number Ψ; let's call this Φ. Next, we again normalize by the sum of all possible values of the phase displacement. To get the first part we take the integral

$$\int_0^\Phi ie^{-\phi^2} d\phi = \left(\frac{i\sqrt{\pi}}{2}\right) \operatorname{erf}(\Phi).$$

To get the normalizer we take the inverse of the integral $\int_0^\infty e^{-\phi^2} d\phi$ to get $\frac{2}{i\sqrt{\pi}}$. The imaginary units again cancel and the product of these gives us

$$\Pr(\psi < \Psi) = \operatorname{erf}(\Phi),$$

which is the basic error function.

The probability that ψ is *at least* some value Ψ is then the basic complementary error function, or

$$\Pr(\psi \geq \Psi) = 1 - \operatorname{erf}(\Phi) = \operatorname{erfc}(\Phi).$$

Notice that in order to get the normalizer here we integrate from zero to infinity and not negative to positive infinity as we did with the Gaussian normalizer Q. This is due to the principle of unitarity. We can integrate from negative to positive infinity to normalize $\Pr(\psi)$ because all such values in that range produce non-negative probability results no greater than unity. But if we integrate over negative values to normalize $\Pr(\psi < \Psi)$ we end up with negative probabilities, which would violate the principle.

But what about the probabilities that the absolute price displacement $|x|$ is less than—or at least—some arbitrary value X? We must first replace Φ with $\sqrt{\mathcal{S}}$ and then square the entire function for the same reasons we discussed in the case of the the probability of $|x|$. This gives us

$$\Pr(|x| < X) = \operatorname{erf}\left(X\sqrt{\frac{m}{2t}}\right)^2, \tag{19}$$

[23] Feynman at 373.

and

$$\Pr(|x| \geq X) = \text{erfc}\left(X\sqrt{\frac{m}{2t}}\right)^2. \qquad (20)$$

This means the price-related elements of this theory expressed as normalized error functions can be thought of as

$$\text{erf}(\sqrt{\mathcal{S}})\,|x < X\rangle + \text{erfc}(\sqrt{\mathcal{S}})\,|x \geq X\rangle$$

since $\text{erf}(\sqrt{\mathcal{S}})^2 + \text{erfc}(\sqrt{\mathcal{S}})^2$ must equal unity. This follows the general Dirac "bra-ket" notation $y\,|Y\rangle + y'\,|Y'\rangle$ for describing states of a system. Here, $|Y\rangle$ and $|Y'\rangle$ are the complementary states "Y" and "not Y," respectively, $|y|^2$ is the probability of finding the state Y in the system, and $|y'|^2$ is the probability of finding the state Y'. As previously stated, $\text{erf}(\sqrt{\mathcal{S}})^2$ is the probability of finding the state $x < X$ for an asset and $\text{erfc}(\sqrt{\mathcal{S}})^2$ is the probability of finding the state $x \geq X$.

This might be merely an interesting aside. Yet, it might also be a catalyst for further discussion on the long-standing question about what probability actually *is*. We often assume that relative frequency is equivalent to probability as the sample approaches the population, but in the infinite case there is no mathematical proof that relative frequency *is* probability, only that it *probably* is. In other words, relative frequency might be only a very close proxy for probability but not actually be probability itself. Without getting more deeply into the debate in this paper, it is possible that probability doesn't just happen to be related to the Lagrangian, but is rather a thing that is categorically defined by it. If this happens to be the case then probability would be defined in the most general terms as *some function of displacement in relative configuration space*. That would be the definition of what probability *is*.

6. Testing the Theory

Now that we've laid the theoretical groundwork, let's see if the theory is, at a minimum, consistent with certain historical asset prices.

A. Extreme Price Displacements

We'll first try to predict the maximum possible price displacement of an asset for a calendar week. As it turns out, the mathematics works if we define our unit of elapsed time as a single trading day. For any calendar year there are 252 trading days in the United States. Given that there are 52 weeks in a calendar year, the average number of units per calendar week is $\frac{252}{52} \approx 4.846$ days. This will be our average value for t.

Let's start with the United States S&P 500 (ticker symbol "SPX").[24] For the period from the week starting 7 January 2007 to the end of the week starting 28 September 2008, $|\bar{x}| =$ 25.72.73. We take $\bar{p}$ from a more granular time frame since we are looking for an approximation related to the asset itself, not merely the asset for a period t. The most granular time frame for which data are freely available are daily data. Using these, we calculate $\bar{p} = 0.5376$.

From Eq. (4) we get an approximate value for $|\mathcal{R}|$ given our sample of 218.37 points.[25] This means the price displacement extreme for any week close to or immediately proceeding

[24]All weekly values of price displacements are measured from the previous week's closing price to the current week's closing price; that is, t_A is the previous week's closing price and t_B is the current week's closing price.

[25]When testing this theory with historical data it is important to remember that p is the relative frequency of x being "up" (positive) and q the relative frequency of x being "down" (negative). This means the relative frequencies must be computed with the definitions $p > 1/2$ and $q < 1/2$. Both must represent non-zero price displacements. For relatively small values of n (e.g., $t = 1$ week) this is not much of a problem as the number of times $x = 0$ is small. Yet, when n is large (e.g., $t = 1$ second) the number of times $x = 0$ can be disproportionately large and will skew the final value of $|\mathcal{R}|$ if the occasions that $x = 0$ are attributed to either p or q.

this sample would be approximately ± 218.37 points.

Looking at the data in this sample period we find that the absolute price displacement extremum for any t was $|x| = 114.04$ points (more precisely, $x = -114.04$). This is far from the projected constraint of ± 218.37 points. Yet, for the week starting 5 October 2008 (the week immediately proceeding the sample), the market crashed causing a one-week price displacement of $x_{\text{SPX}} = -200.01$. This came close to, but as theorized did not exceed, the minimum constraint of -218.37 points. What's interesting is that we calculated this extremum from our equations based on historical data from *before* the actual crash.

If we look at the same periods for the Dow Jones Industrial Average (ticker symbol "DJI") we see that for the same period we can approximate weekly price displacement extrema of ± 2056.20. The actual price displacement for the week starting 5 October 2008 (the crash week) was -1874.19 points. Again, the price displacement was very close to the projected extreme without exceeding it.

B. Probability Calculations

Our first probability question to test becomes: "What is the probability that the absolute price displacement is equal to some arbitrary value X, or $\Pr(|x| = X)$?" Recall that the appropriate equation to answer this question is

$$\Pr(|x| = X) = \left(\frac{m}{2\pi t}\right) e^{-\frac{mX^2}{t}}.$$

These equations can also be tested with assets such as SPX and DJI. If we set our constant X equal to the modulus of the price displacement we find from historical data (i.e., X becomes the actual absolute price displacement value for some specified period t) and compare the probabilities calculated by this theory with those calculated by relative frequencies over multiple years of historical data we find that deviations are ≤ 0.001.

Our next probability question to test is: "What is the probability that the absolute price displacement is *at least* some value X?" In other words, what is the solution to the probability function $\Pr(|x| \geq X)$? Again, recall that the appropriate equation is

$$\Pr(|x| \geq X) = \operatorname{erfc}\left(X\sqrt{\frac{m}{2t}}\right)^2.$$

If we again set X equal to an historical price displacement value for some specified period t we should be able to check the accuracy of this postulate against the same relative frequency measure based on historical data. The square of the complementary error function yields predicted probabilities that are < 0.02 points different than the relative frequencies taken from the historical sample. Since $\Pr(|x| \geq X) = 1 - \Pr(|x| < X)$, we see that the test produces identical results for

$$\Pr(|x| < X) = \operatorname{erf}\left(X\sqrt{\frac{m}{2t}}\right)^2.$$

These results hold for both of our test assets SPX and DJI. The significance is that we are able to approximate probabilities not from historical data, but from measures of physical attributes, or at least their analogs; namely, mass, time, and distance.

C. Calculating Inertial Coefficients

We can also approximate the measure of the inertial coefficient of each asset SPX and DJI with this information. For the asset SPX where $|\mathcal{R}| \approx 218.37$ points for t, the inertial coefficient is approximately 2.006×10^{-3}. For DJI where $|\mathcal{R}| \approx 2056.20$ points for t, the inertial coefficient is approximately 2.263×10^{-5}.

We can compare the approximations we get from the method just described with another method involving the manipulation of our probability equations. If we manipulate Eq. (20) to solve for the inertial coefficient we get

$$m = 2t\left(\frac{\operatorname{erfc}^{-1}\sqrt{\Pr(|x| \geq X)}}{x}\right)^2. \qquad (21)$$

Using the same sample period and t-value for SPX as before we find that a value for m that equates the average predicted probabilities and the average actual probabilities for the sample becomes approximately 1.915×10^{-3}. Working with the equations already discussed we get an approximate absolute price displacement extreme of $|\mathcal{R}| \approx 223.50$. If instead we find a value for m that minimizes the residual sum of squares for the predicted and actual probabilities we get an inertial coefficient of approximately 1.904×10^{-3} and an approximate absolute price displacement extreme of $|\mathcal{R}| \approx 224.15$. We find similar results with DJI.

All of our approximation methods give results very close to each other, suggesting that at least the theory is not inconsistent with itself. Note that the asset with the greater inertial coefficient (SPX) had a less extreme price displacement during the crash week of 5 October 2008. For physical phenomena, one would expect an object with a greater inertial coefficient to move a lesser distance than one with a lower inertial coefficient, *ceteris paribus*. It appears that financial assets react similarly if one regards movement to be through a dimension of price instead of space.

7. Conclusions

One way to summarize this approach is as follows: Asset prices move either up or down during some period of elapsed time. This up-and-down motion, by definition, is linear for the specified period. Linear motion in one dimension, in this case the dimension of price, is the result we would expect if the asset's mechanics comply with the principle of stationary action. Yet, linear motion in one dimension is also achievable through circular motion in two dimensions. If the single dimension we observe is "real" and the unobserved second dimension "imaginary," then circular motion in the complex plane can explain the observable linear motion of assets through the price dimension.

Given this construct, the phase of the complex wave function and the phase of its complex conjugate are equally likely to produce an observable price displacement. The square of the wave's phase displacement is then responsible for any observable linear motion, and therefore, the stationary action. As a result, we hold that the square of the wave function's phase displacement is equal to the action of the asset. Applying the action principle to defined periods produces price displacement results for assets that are consistent with historical price data for those periods.

What are some of the implications of this theory and the preliminary results we've seen so far? The first is that asset price displacements might comply with certain physical laws. We showed here theoretically, and the historical data do not contradict the conclusion, that asset price displacements are perhaps constrained by extreme positive and negative values beyond which whey cannot go for specified elapsed periods of time.

Might this theoretical discovery act as a sort of "Black Swan" predictor, at least in magnitude?[26] While there is nothing in this theory that would tell us *when* a low-probability event would occur such as the market crash during the week of 5 October 2008, the theory did accurately predict the *magnitude* of the constraint of the price displacement for any trading week, including the week of the crash.

Another way for researchers to test this the-

[26] Nassim Nicholas Taleb, The Black Swan: The Impact of the Highly Improbable (2d ed. 2010).

ory is to look at the correlation between price displacements and the net trading volume for appropriate samples of periods t. If one examines the correlation between the total trading volume and the price displacement for the sample one should find little to no correlation. There should be a sense of randomness. Yet, if one examines the net trading volume (i.e., the volume attributed to α less the volume attributed to β) one should find a fairly strong correlation. My guess is that the strongest correlation will be nonlinear.

Lastly, even though these results seem promising, we should remember that the menu is not the meal and the map is not the terrain. Simply because the results of employing such a model suggest that markets comply with the physical laws implicit in the model does not mean that this is an exact explanation of phenomena we witness in everyday market mechanics; that is, these very well might not be the "actual descriptions of the forces and interactions at hand."[27] Furthermore, this theory might only explain macro-market movements and might not be as applicable to individual assets. Macro-market mechanics are captured, to varying degrees, in market indices such as the American S&P 500, Dow 30, NASDAQ 100, and Russell 2000, Japan's Nikkei 225, Britain's FTSE 100, China's A50, Germany's DAX, France's CAC 40, the Euro Zone's Euro Stoxx 50, and Poland's WIG 20. Yet, individual assets—even those composing these stock indices—might not have the same mechanical patterns. Again, further research is need to find the limits of this theory's application.

This theory is not without assumptions, but it is my hope that the assumptions it has are at least intuitive or, better still, conform with historical data. Instead of looking at excessively complex possibilities, I've tried very hard throughout to focus on what works and then seek to explain it as simply as possible, but no simpler.[28] I leave it to the reader to judge to what extent I have succeeded or failed in this objective.

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[27] Mauro Gallegati, Steve Keen, Thomas Lux & Paul Ormerod, *Worrying Trends in Econophysics*, 370 Physica A 1, 4 (2006).

[28] *See* Albert Einstein, *On the Method of Theoretical Physics*, 1 Philo. of Sci. 163, 165 (1934) ("It can scarcely be denied that the supreme goal of all theory is to make the irreducible basic elements as simple and as few as possible without having to surrender the adequate representation of a single datum of experience.").