# Universality of market superstatistics


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## ABSTRACT

We use a continuous-time random walk (CTRW) to model market fluctuation data from times when traders experience excessive losses or excessive profits. We analytically derive „superstatistics" that accurately model empirical market activity data (supplied by Bogachev, Ludescher, Tsallis, and Bunde) that exhibit transition thresholds. We measure the interevent times between excessive losses and excessive profits, and use the mean interevent time as a control variable to derive a universal description of empirical data collapse. Our superstatistic value is a power-law corrected by the lower incomplete gamma function, which asymptotically tends toward robustness but initially gives an exponential. We find that the scaling shape exponent that drives our superstatistics subordinates itself and a „superscaling" conguration emerges. We use superstatistics to describe the hierarchical activity which reproduces the negative feedback. Our results indicate that there is a functional (but not literal) balance between excessive profits and excessive losses that can be described using the same body of superstatistics, but different calibration values and driving parameters.


## Principal goal

Empirical market data on excessive profits and losses [1–4] define excessive profits as those greater than some positive fixed threshold $Q$ and excessive losses as those below some negative threshold $-Q$. The mean interevent time between losses $R_Q$ vs. $Q$ has been used as an aggregated basic variable.

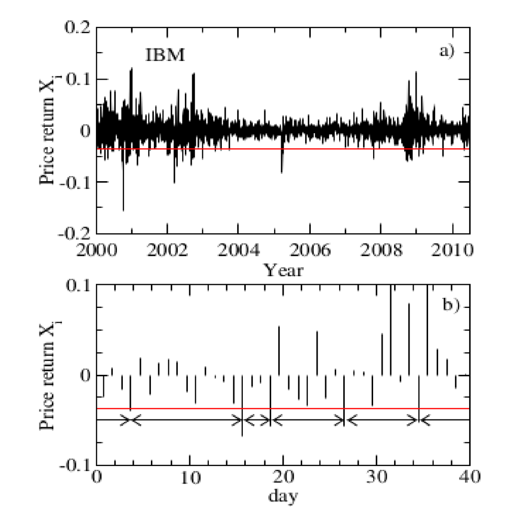


Fig. 1: (Colour on-line) Illustration of the relative daily price returns $X_i$ of the IBM stock between (a) January 2000 and June 2010 and (b) August 27 and October 23, 2002. The red line shows the threshold $Q \simeq -0.037$, which corresponds to an average interoccurrence time of $R_Q = 70$. In (b) the interoccurrence times are indicated by arrows.

Taken from [1].

Our principal goal is to model the empirical data associated with single-variable statistics, i.e., (i) the mean interevent time period $R_Q$ between extreme (excessive) losses, defined as those below a negative threshold $-Q$, as a function of the $Q$ (> 0) value and (ii) the distribution $\psi_Q(\Delta_Q t)$ of interevent times between losses $\Delta_Q t$, previously described using *ad hoc* q-exponentials [1, 2].

## Mean interevent time

Mean interevent time $R_Q$ between excessive losses fulfils:

$$R_Q^{-1} = P(-\varepsilon \leqslant -Q) = P(\varepsilon \geqslant Q) = \int_Q^{\infty} D(\varepsilon)\, d\varepsilon, \qquad (1)$$

where $D(\varepsilon)$ is the density of returns given by the Weibull distribution of extreme losses [5, 6, 7],

$$D(\varepsilon) = \frac{\eta}{\bar{\varepsilon}} \left(\frac{\varepsilon}{\bar{\varepsilon}}\right)^{\eta - 1} \exp\left(-\left(\frac{\varepsilon}{\bar{\varepsilon}}\right)^{\eta}\right), \ \bar{\varepsilon}, \eta > 0. \qquad (2)$$

Substituting (2) into (1), we obtain

$$R_Q = \exp\left(\left(\frac{Q}{\bar{\varepsilon}}\right)^{\eta}\right). \qquad (3)$$

## Superstatistics – assumptions

Unconditional distribution $\psi_Q(\Delta_Q t)$ of the interevent times:

$$\psi_Q(\Delta_Q t) = \int_Q^{\infty} \psi_Q(\Delta_Q t|\varepsilon) D(\varepsilon)\, d\varepsilon. \qquad (4)$$

We assume the conditional distribution $\psi_Q(\Delta_Q t|\varepsilon)$ in the form

$$\psi_Q(\Delta_Q t|\varepsilon) = \frac{1}{\tau_Q(\varepsilon)} \exp\left(-\frac{\Delta_Q t}{\tau_Q(\varepsilon)}\right) \qquad (5)$$

and the relaxation time as stretched exponential function,

$$\tau_Q(\varepsilon) = \tau_Q(0) \exp\left((B_Q \varepsilon)^{\eta}\right). \qquad (6)$$

## Superstatistics and superscaling forms

Substituting (6) and (5) into (4) we finally obtain

$$\psi_Q(\Delta_Q t) = \frac{1}{\tau_Q(Q)} \frac{\alpha_Q}{\left(\Delta_Q t/\tau_Q(Q)\right)^{1+\alpha_Q}} \times \gamma_{Euler}\left(1+\alpha_Q, \Delta_Q t/\tau_Q(Q)\right), \qquad (7)$$

where $\gamma_{Euler}$ is the lower incomplete gamma function[2] and

$$\alpha_Q = \frac{1}{(B_Q \bar{\varepsilon})^{\eta}} = \frac{1}{\ln\left(\tau_Q(\bar{\varepsilon})/\tau_Q(0)\right)} \qquad (8)$$

is the scaling exponent. We proved superscaling of $\ln R_Q$:

$$\frac{1}{\alpha_Q} = B \ln^{\zeta} R_Q, \qquad (9)$$

where $B$, $\zeta$ (> 0) are $Q$-independent control parameters.

## Empirical verification of our formulas

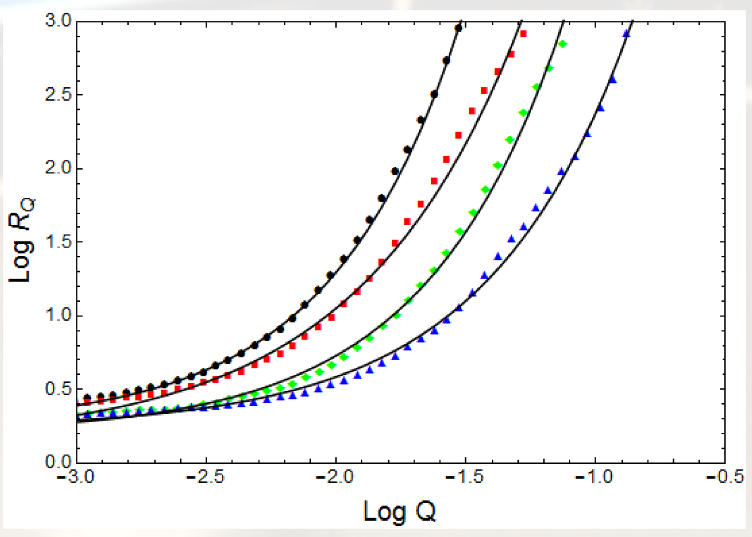


Fig. 2. Mean interevent time period $R_Q$ vs. threshold $Q$ for four typical classes of quotations. Black circles, red squares, green rhomboids, and blue triangles concern USD/GBP exchange rate, S&P 500 index, IBM stock, and WTI (the crude oil) empirical data (from January 2000 to June 2010), respectively, taken from Fig. 2 in ref. [1] (plotted from the top curve down to the bottom one). The solid, well fitted curves present predictions of our formula (3). Subtle wavy deviations from these predictions are not considered in this work. (Empirical data were used by permission of the EPL.)

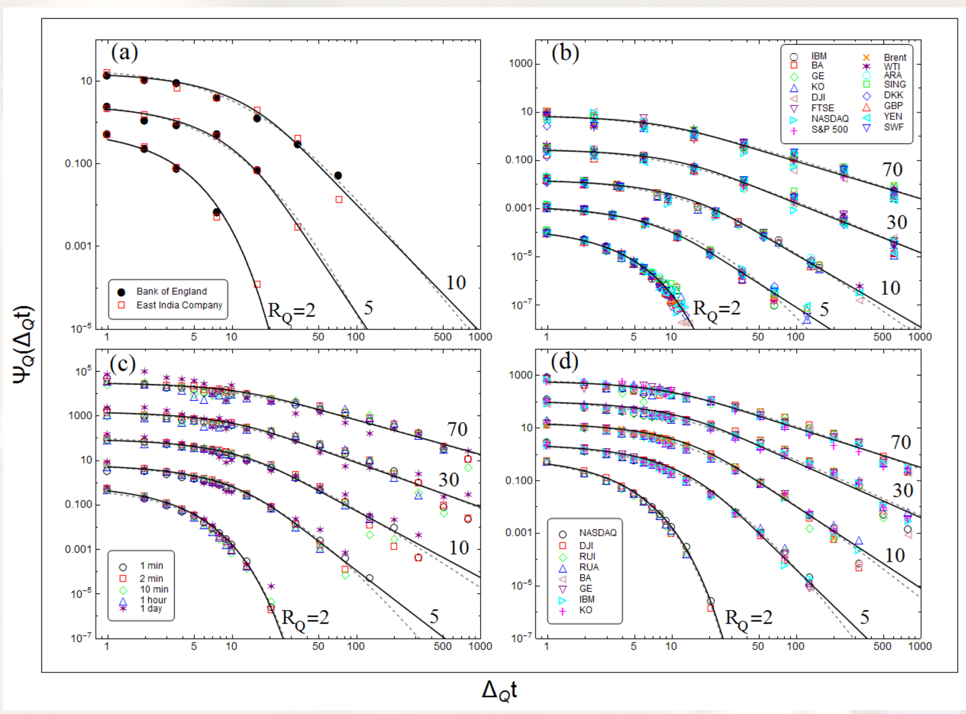


Fig. 3. Collected plots of empirical distributions (colored marks) and theoretical superstatistics, $\psi_Q(\Delta_Q t)$, (black solid curves), which are predictions of our formula (7) (while the dashed curves were given by q-exponential shown by eq. (3) in [1]) vs. interevent time, $\Delta_Q t$, for the monthly returns (a), for the relative daily price returns for sixteen typical examples of financial data in the period 1962-2010 (b), from minutes over the hours to daily returns for NASDAQ between March 16, 2004 and June, 2006 (c), and for the detrended minute-by-minute eight most typical examples of financial data (d). (All empirical data were drawn from [1, 2] for permission of EPL and PRE.)

## CONCLUSIONS

We find an explicit closed form of the threshold interevent time superstatistics (7) that is valid for excessive losses, is the foundation of the continuous time random walk (CTRW) formalism, and that is useful in the study of a double action market (see [8] and refs. therein). These superstatistics are more credible than than the q-exponential distribution that is applied *ad hoc* in this context in [1, 2], and they agree with the key empirical relation between the mean interevent time $R_Q$ and the threshold $Q$ (see Fig. 2).


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[2] When the clustering phenomenon occurs, i.e., when the relaxation time $\tau'_Q(\varepsilon) = \tau_Q(0) \exp(-(B_Q \varepsilon)^{\eta})$, the final result is analogous, but with upper (instead of lower) incomplete gamma function.


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