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Student's t-distribution *versus* Zeldovich-Kompaneets solution of diffusion problem

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Student's t-distribution has the probability density function

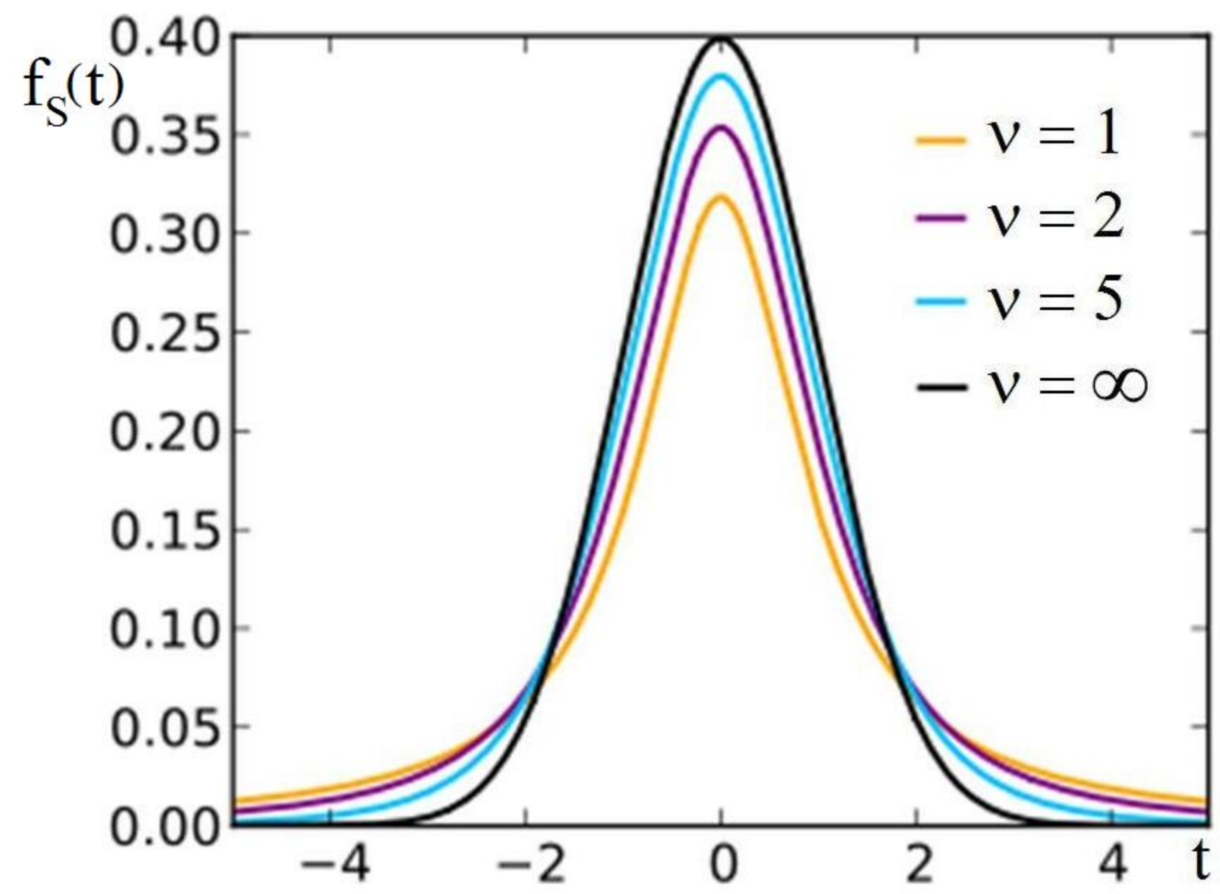
$$f_S(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{t^2}{\nu}\right)^{-(\nu+1)/2}$$

Here ν is known as the number of degree of freedom (df) of the distribution and Γ is the Gamma function. The variance is given for $\nu > 2$ by

$$\sigma^2 = \int_{-\infty}^{\infty} t^2 f_S(t)dt = \frac{\nu}{\nu-2}$$

The kurtosis

$$\kappa = \left(\frac{1}{\sigma^2}\right)^2 \int_{-\infty}^{\infty} t^4 f_S(t)dt = 3\frac{\nu-2}{\nu-4} > 3 \quad \text{for} \quad \nu > 4$$



Student's t-distribution for different degrees of freedom ν.

Let j denote a flux of Brownian particle diffusion. According to Fick's law $j = D\,\partial f/\partial x$, where $f = f(x, t)$ is the distribution function, which depends on position x and time t. A nonlinear process in which the diffusion coefficient takes the form

$$D = a f^{-n},\ a = a(t) > 0,\ \ n > 0$$

is considered.

(Super) Diffusion equation

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial f}{\partial x}\right) \quad \text{or} \quad \frac{\partial f}{\partial t} = a\frac{\partial}{\partial x}\left(f^{-n}\frac{\partial f}{\partial x}\right)$$

$$\xi = \frac{x}{A^{\frac{1}{2-n}}}$$

$$\frac{\partial}{\partial \xi}\left[(2-n)\varphi^{-n}\frac{\partial \varphi}{\partial \xi} + \xi\varphi\right] = 0$$

$$A = A(t) = \int_0^t a(\tau)d\tau$$

$$f = \frac{1}{A^{\frac{1}{2-n}}}\varphi(\xi)$$

$$\varphi = \varphi(\xi) = \left[\frac{n}{2(2-n)}\left(\xi_0^2+\xi^2\right)\right]^{-1/n}$$

$$f = \left\{2\left(\frac{2}{n}-1\right)A\,B^2\left(\frac{1}{n}-\frac{1}{2},\frac{1}{2}\right)\right\}^{-1/(2-n)}\left(1+\frac{x^2}{x_0^2}\right)^{-1/n}$$

$$x_0 = A^{\frac{1}{2-n}}\,\xi_0$$

$$B\left(\frac{1}{n}-\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma(\frac{1}{n}-\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{n})}$$

$$\xi_0^{2-n} = 2\left(\frac{2}{n}-1\right)B^n\left(\frac{1}{n}-\frac{1}{2},\frac{1}{2}\right)$$

$$f(x,0) = \delta(x)$$

The variance

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 f dx = \frac{n}{2-3n}x_0^2$$

The kurtosis

$$\kappa = \left(\frac{1}{\sigma^2}\right)^2 \int_{\infty}^{\infty} x^4 f dx = 3\,\frac{2-3n}{2-5n}$$

The substitution

$$n = \frac{2}{\nu+1}$$

gives the value of kurtosis for the Student's t-distribution.

References

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