Bayesian Value-at-Risk and Expected Shortfall for a portfolio (multi- and univariate approaches)

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Plan of the presentation

1. Introduction
2. Portfolio VaR and Expected shortfall – concepts, notation, modelling approaches
3. Foundations of Bayesian VaR and ES assessments
4. Bayesian *n*-variate MSF – scalar BEKK(1,1) volatility model
5. Non-parametric models: CAViaR, CARE
6. Empirical examples (VaR and ES for portfolios with 50 assets)
7. Concluding remarks

1. Introduction – motivation and contents of the paper

- VaR and ES are purely probabilistic concepts related to the left or right tails of the distribution of the future portfolio value; in practice they are based on statistical models estimated using historical data on asset prices
- Bayesian approach can formally unify the theory and practice of VaR and ES (through the predictive distribution of future returns); in particular, it properly takes into account parameter uncertainty and deals with non-linearity between logarithmic returns and VaR

- Bayesian VaR and ES for large portfolios require simple multivariate volatility models that would have proper tail behaviour
- Osiewalski [2009] built hybrid MSF-SBEKK models which can be used for large portfolios
- VaR and ES require only the distribution of the future value of the portfolio, and this can be derived using a univariate model for the historical values of this portfolio.
- univariate approach is much simpler, but it ignores the covariance structure of the assets.

The aim of the paper:

- compare the n-variate and univariate approaches to risk assessment for a large portfolio
- show how the new hybrid MSF–SBEKK specification works in practice
- show the merits of the Bayesian parametric approach to Value-at-Risk and Expected Shortfall modelling

2. Portfolio VaR and ES – concepts, notation

The *s* period return rate on the portfolio:

$$R^{*}_{t|t+s} = (W_{t+s} - W_t)/W_t = \sum_{i=1}^{n} \omega_{t,i} R_{t|t+s,i} \,,$$

where

$W_t = \sum_{i=1}^{n} a_i S_{t,i}$ - the time *t* value of the portfolio,

$S_{t,i}$ - the price of asset *i* at time *t*, a_i - the number of units of asset *i*.

$R_{t|t+s,i} = (S_{t+s,i} - S_{t,i})/S_{t,i}$ - the *s* period return rate on asset *i*,

$\omega_{t,i} = a_i S_{t,i}/W_t$ - the share of asset *i* in the time *t* porfolio value.

Value at Risk for a long trading position

$VaR^{l}_{T|T+s}(\alpha)$ for a given probability level α is defined by the following equality:

$$\Pr\{W_{T+s} \le W_T - VaR^{l}_{T|T+s}(\alpha) \mid \psi_T\} = \alpha,$$

which can be written as

$$\Pr\{R^{*}_{T|T+s} \le -VaR^{l}_{T|T+s}(\alpha)/W_T \mid \psi_T\} = \alpha,$$

ψ_T - the whole past of the observed asset prices

The relative VaR is the α-quantile of a non-linear function of future logarithmic returns:

$$\Pr\{-1+\sum_{i=1}^{n}\omega_{T,i}\exp(\sum_{j=1}^{s}r_{T+j,i})\leq -VaR_{T|T+s}^{l}(\alpha)/W_T \mid \psi_T\}=\alpha$$

where

$$r_{t+j,i}=\ln(S_{t+j,i}/S_{t+j-1,i})=\ln(R_{t+j-1|t+j,i}+1),$$

Consequences of linear approximation:

$$\exp(\sum_{j=1}^{s} r_{t+j,i}) \approx 1 + \sum_{j=1}^{s} r_{t+j,i}$$

- can lead to serious errors, especially when *s* is so large that the *s* period ahead return distribution is diffuse.

Consider an example with one asset (*n*=1) and Student *t* distribution, *St*(4), for $10\, r_{T|T+s}$. Then $\Pr\{St(4) \le 10 \ln[1-(VaR^{l}_{T|T+s}(\alpha)/W_T)]\} = \alpha$.

Table 1. Relative VaR for $r_{T|T+s}$ distributed as 0.1 *St*(4)

α	0.005	0.01	0.0125	0.025	0.05
approximate VaR	0.4604	0.3747	0.3495	0.2776	0.2132
true VaR	0.3690	0.3125	0.2950	0.2424	0.1920

For small α, the true relative VaR can be overestimated quite substantially.

Value at Risk for a short trading position

$VaR^{s}_{T|T+s}(\alpha)$ for a given probability level α is defined by the following equality:

$$\Pr\{W_{T+s} \geq W_T + VaR^{s}_{T|T+s}(\alpha) \mid \psi_T\} = \alpha,$$

which can be written as

$$\Pr\{R^{*}_{T|T+s} \geq VaR^{s}_{T|T+s}(\alpha)/W_T \mid \psi_T\} = \alpha,$$

$$\Pr\{-1 + \sum_{i=1}^{n} \omega_{T,i} \exp(\sum_{j=1}^{s} r_{T+j,i}) \geq VaR^{s}_{T|T+s}(\alpha)/W_T \mid \psi_T\} = \alpha$$

Disadvantages of the VaR:

- the VaR does not tell anything about the potential size of loss that exceeds the VaR level

- the VaR is not a *coherent* measure (the VaR lacks the property of *sub-additivity*, e.g. the VaR of a portfolio can exceed the sum of the VAR levels of its sub-portfolios)

see Hoogerheide and van Dijk [2008]

Expected Shortfall for long and short trading positions

long trading position:

$$ES^{l}_{T|T+s}(\alpha) = -E_T(W_{T+s} - W_T \mid W_{T+s} \leq W_T - VaR^{l}_{T|T+s}(\alpha)),$$

short trading position:

$$ES^{s}_{T|T+s}(\alpha) = E_T(W_{T+s} - W_T \mid W_{T+s} \geq W_T + VaR^{s}_{T|T+s}(\alpha))$$

- it says something about losses exceeding the VaR level
- ES is a *sub-additive*, coherent measure (the ES of a portfolio can not exceed the sum of the ES measures of its sub-portfolios, Hoogerheide and van Dijk 2008)

3. Foundations of Bayesian VaR and ES assessments

Bayesian inference relies on the following decomposition of the joint density:

$$p(y, y_f, \theta) = p(y_f \mid y, \theta)\, p(y \mid \theta)\, p(\theta) = p(y_f \mid y, \theta)\, p(\theta \mid y)\, p(y),$$

where y represents observed return rates, y_f denotes unobserved returns, θ is vector of parameters and latent variables.

Inference on all unknown and unobserved quantities (parameters, latent variables and future observables) can be based on the joint posterior – predictive density function:

$$p(\theta, y_f \mid y) = p(y_f \mid y, \theta)\, p(\theta \mid y),$$

where

$p(y_f \mid y, \theta)$ - the sampling predictive density (conditional on the parameters and latent variables),

$p(\theta \mid y) = p(y \mid \theta)\, p(\theta) / p(y)$ - the posterior density (of the parameters and latent variables),

$p(y) = \int_{\Theta} p(y \mid \theta)\, p(\theta)\, d\theta$ - the marginal density of the observed returns.

If we are only interested in prediction of future returns, as in the case of determining the portfolio VaR, we use the Bayesian predictive distribution:

$$p(y_f \mid y) = \int_{\Theta} p(y_f \mid y, \theta)\, p(\theta \mid y)\, d\theta,$$

which fully reflects uncertainty regarding θ, given the data, the choice of a sampling model and a prior density; this uncertainty is formalized through the posterior density. If a particular function of y_f is of interest (like $R^*_{T|T+s}$, the s period ahead portfolio return), its distribution can be obtained from $p(y_f \mid y)$.

Non-Bayesian forecast of $R^*_{T|T+s}$ is based on $p(y_f \,|\, y, \theta = \hat{\theta})$ with (e.g.) thinner tails (!)

consider *n*=1, $p(r_{T|T+s} \,|\, y, \theta)$ is N(0, τ), τ has Gamma(v/2, v/2) posterior; then $p(r_{T|T+s} \,|\, y)$ is St(v)

In this case the usual non-Bayesian VaR would be calculated using the thin Normal tail and the Bayesian VaR would be based on the thicker Student tail, properly reflecting parameter uncertainty. Of course, there is little practical difference between both approaches when τ is estimated very precisely (large ν), but this need not be the case (like when v is small, which leads to substantial differences).

4. Bayesian *n*-variate MSF – scalar BEKK(1,1) volatility model

Let $r_t = (r_{t,1} \ldots r_{t,n})$ denote *n*-variate observations on logarithimc return rates, which we model using the basic VAR(1) framework:

$$r_t = \delta_0 + r_{t-1}\Delta + \varepsilon_t, \quad t=1,\ldots,T,\ldots T+s.$$

$\{\varepsilon_t\}$ – MSF-SBEKK process

MSF-SBEKK process (see Osiewalski 2009, Osiewalski and Pajor 2009):

$$\varepsilon_t = \zeta_t H_t^{1/2} \sqrt{g_t}, \qquad \ln g_t = \phi \ln g_{t-1} + \sigma_g \eta_t,$$

$$H_t = (1-\beta-\gamma)A + \beta(\varepsilon_{t-1}'\varepsilon_{t-1}) + \gamma H_{t-1},$$

$$(\zeta_t \quad \eta_t)' \sim iiN(0_{[(n+1)\times 1]}, I_{n+1})$$

Here:

$\{g_t\}$ is a latent process as in the MSF model,

H_t is a square matrix of order n that has the scalar BEKK(1,1) structure

(when $\sigma_g \to 0$ and ϕ=0 we are in the SBEKK model, while $\beta = 0$ and $\gamma = 0$ lead to the MSF case)

Prior distribution:

$\delta = (\delta_0 \ (\text{vec } \Delta)')' \sim N(0, I_{n(n+1)})$, (truncated by the restriction that all eigenvalues of Δ lie inside the unit circle)

$A^{-1} \sim IW$ with mean I_n,

$H_0 = h_0 I_n$, $h_0 \sim Exp(1)$, $\qquad (\beta, \gamma) \sim 0.5\ I_{\{(b,\, c):\ b+c < 1\}}(\beta, \gamma)$

$\phi \sim N(0, 100)\ I_{(-1,1)}(\phi)$, $\qquad \sigma_g^{-2} \sim Exp(1/200)$

$\ln(g_0) = 0$.

In practice: for A and δ we use the approximation explained in Osiewalski and Pajor [2009], OLS.

5. Non-parametric models: CAViaR, CARE

For the sake of comparison we also use non-parametric models:

- **the Conditional Autoregressive Value at Risk (CAViaR) model** (with asymmetric slope):

$$q_t(\alpha) = \beta_0 + \beta_1 q_{t-1}(\alpha) + \beta_2 \mid D_{t-2|t-1} \mid + \beta_3 \mid D_{t-2|t-1} \mid I_{(-\infty,0)}(D_{t-2|t-1})$$

of Engle and Manganelli (2004); it is applied directly to the series $\{D_{t|t+s}\}$ of daily value changes $D_{t|t+s} = W_{t+s} - W_t$, s = 1 (not to the logarithmic returns).

- **the Conditional Autoregressive Expectiles (CARE) model**:

$$\mu_t(\tau) = \gamma_0 + \gamma_1 \mu_{t-1}(\tau) + \gamma_2 \mid D_{t-2|t-1} \mid + \gamma_3 \mid D_{t-2|t-1} \mid I_{(-\infty,0)}(D_{t-2|t-1})$$

where $\mu_t(\tau)$ is the conditional τ expectile,

$$\mu_t(\tau) = \arg\min_{\mu} E_{t-1}\left(\mid \tau - I_{(-\infty,0)}(D_{t-1|t} - \mu) \mid (D_{t-1|t} - \mu)^2 \right).$$

The best choice of γ for a given τ is:

$$\hat{\gamma} = \arg\min_{\gamma} \sum_{t=1}^{T} \mid \tau - I_{(-\infty,0)}(D_{t-1|t} - \mu_t(\tau) \mid (D_{t-1|t} - \mu_t(\tau))^2 ,$$

where γ is a vector of parameters in the CARE model.

A link between expectiles and ES:

$$ES_t^l(\alpha) = -\left(1 + \frac{\tau}{(1-2\tau)\alpha}\right)\mu_t(\tau) + \left(\frac{\tau}{(1-2\tau)\alpha}\right)E_{t-1}(D_{t-1|t}),$$

see Taylor [2008].

Consequently, we obtain the **Conditional Autoregressive ES model:**

$$ES_t^l(\alpha) = \lambda_0 + \lambda_1 ES_{t-1}^l(\alpha) + \lambda_2 \,|D_{t-2|t-1}| + \lambda_3 \,|D_{t-2|t-1}|\, I_{(-\infty,0)}(D_{t-2|t-1}),$$

when $E_{t-1}(D_{t-1|t}) = 0,\ \lambda_1 = \gamma_1,\ \lambda_i = -\left(1 + \frac{\tau}{(1-2\tau)\alpha}\right)\gamma_i,\ i = 0, 2, 3.$

6. Empirical examples: VaR and ES for portfolios with 50 assets

As the dataset we use the stock data representing 50 companies (included in mWIG40 and in WIG20), which we used in Osiewalski and Pajor [2009].

We start with $T = 998$ initial observations (covering the period 13.05.2005 – 12.05.2009) and consider $p = 200$ VaR and ES assessments for 1-, 2-, …, 10-day trading horizons.

For Bayesian estimation the whole dataset available at time $T+k$ ($k = 0, 2, \ldots, p\text{-}1$) is used.

We calculate predictive distributions of r_t based on dataset available at time $T+k$ for each $k = 0, 2, \ldots, p\text{-}1$ (up to $T+p\text{-}1 = 1138$). Thus we obtained 200 predictive distributions for 1-, 2-, …, 10-day forecast horizons, and then

$\mathrm{VaR}_{t|t+s}(\alpha)$ for $t = T, \ldots, T + p\text{-}1$, $s = 1, 2, \ldots, 10$.

Figure 1. Daily changes in the portfolio value (a_t = (1, …, 1)′, May 13, 2005 – February 23, 2010).

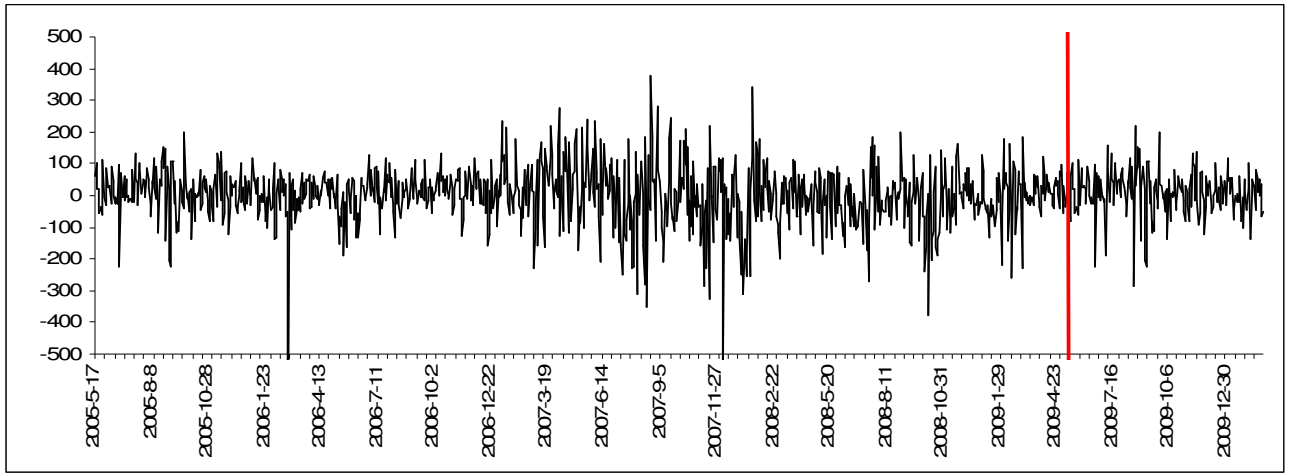


Figure 2. Daily changes in the portfolio value (ω_τ = 1/50, May 13, 2005 – February 23, 2010)

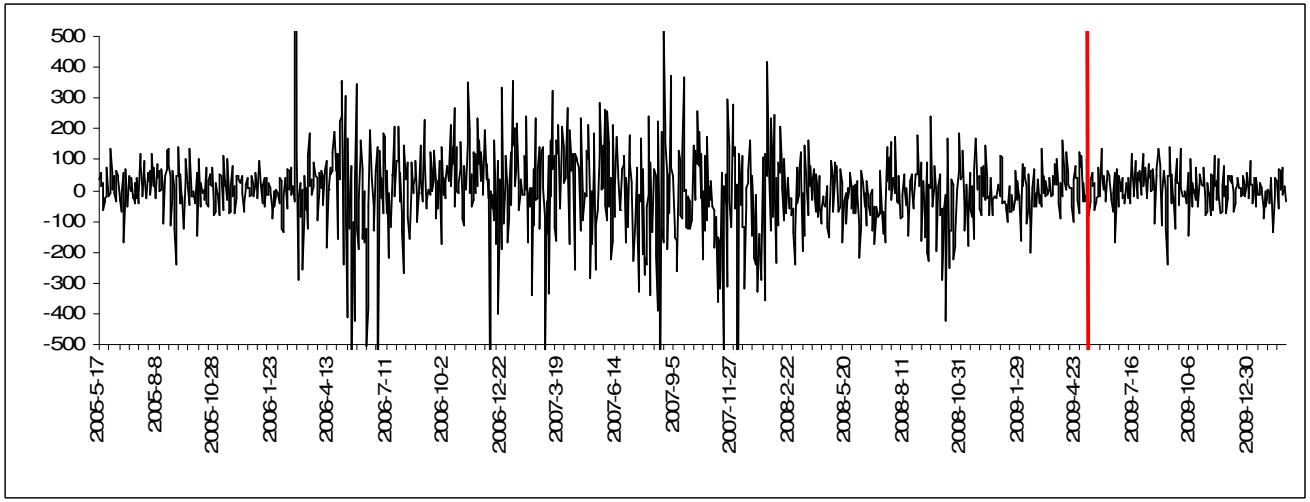


Table 1. The failure rate for $VaR^{l}_{t|t+1}(\alpha)$ and $VaR^{s}_{t|t+1}(\alpha)$

α	Portfolio with a = (1, ..., 1)′			Portfolio with $\omega_{\tau i}$ = 1/50		
	n-variate MSF-SBEKK	univariate MSF-SBEKK	CAViaR	*n*-variate MSF-SBEKK	univariate MSF-SBEKK	CAViaR
Long trading position						
0.01	0.030	0.010	0.015	0.025	0.015	0.01
0.025	0.050	0.030	0.020	0.06	0.02	0.03
0.05	0.100	0.055	0.055	0.09	0.04	0.04
0.1	0.150	0.095	0.105	0.135	0.08	0.095
Short trading position						
0.1	0.205	0.125	0.11	0.195	0.09	0.08
0.05	0.115	0.075	0.07	0.12	0.04	0.03
0.025	0.085	0.04	0.04	0.07	0.015	0.005
0.01	0.045	0.01	0.005	0.025	0.05	0

The failure rate is defined as the proportion of $D_{t|t+1}$'s smaller than the $-VaR^{l}_{t|t+1}(\alpha)$ or the proportion of $D_{t|t+1}$'s bigger than the $VaR^{s}_{t|t+1}(\alpha)$

Table 2. The p-value for the Kupiec test for $VaR^{l}_{t|t+1}(\alpha)$ and $VaR^{s}_{t|t+1}(\alpha)$

α	Portfolio with a = (1, ..., 1)′			Portfolio with $\omega_{\tau i}$ = 1/50		
	n-variate MSF-SBEKK	univariate MSF-SBEKK	CAViaR	*n*-variate MSF-SBEKK	univariate MSF-SBEKK	CAViaR
Long trading position						
0.01	0.022	1.000	0.508	0.073	0.508	1.000
0.025	0.007	0.660	0.639	0.007	0.639	0.660
0.05	0.002	0.749	0.749	0.019	0.502	0.502
0.1	0.045	0.812	0.815	0.115	0.330	0.812
Short trading position						
0.1	0.000	0.255	0.642	0.000	0.632	0.330
0.05	0.000	0.130	0.220	0.000	0.502	0.162
0.025	0.000	0.211	0.211	0.001	0.328	0.027
0.01	0.000	1.000	0.432	0.073	0.432	--

The losses are calculated as

$$L_s = \frac{1}{p} \sum_{t=T}^{T+p-1} l_{t|t+s},$$

- the Lopez loss:

$$l_{t|t+s} = \begin{cases} 1 + (D_{t|t+s} + VaR^l_{t|t+s}(\alpha))^2, & if \quad D_{t|t+s} < -VaR^l_{t|t+s}(\alpha), \\ 0, & if \quad D_{t|t+s} \geq -VaR^l_{t|t+s}(\alpha); \end{cases}$$

- the firm's loss:

$$l_{t|t+s} = \begin{cases} (D_{t|t+s} + VaR^l_{t|t+s}(\alpha))^2, & if \quad D_{t|t+s} < -VaR^l_{t|t+s}(\alpha), \\ cVaR^l_{t|t+s}(\alpha), & if \quad D_{t|t+s} \geq -VaR^l_{t|t+s}(\alpha), \end{cases}$$

Table 3. Lopez loss function for $VaR^{l}_{t|t+1}(\alpha)$ and $VaR^{s}_{t|t+1}(\alpha)$

α	Portfolio with a = (1, ..., 1)′			Portfolio with $\omega_{\tau i}$ = 1/50		
	n-variate MSF-SBEKK	univariate MSF-SBEKK	CAViaR	*n*-variate MSF-SBEKK	univariate MSF-SBEKK	CAViaR
	Long trading position					
0.01	133.304	50.492	**20.475**	55.278	5.554	**4.914**
0.025	221.992	101.562	**70.786**	124.059	43.682	**42.566**
0.05	350.451	193.607	**186.742**	229.571	108.817	**103.296**
0.1	569.952	**372.444**	384.843	405.383	**234.289**	316.663
	Short trading position					
0.1	536.443	257.276	**155.605**	286.866	117.592	**74.442**
0.05	289.704	89.781	**53.788**	114.531	34.746	**14.116**
0.025	144.837	24.875	**16.642**	35.834	9.362	**0.026**
0.01	44.144	1.764	**1.543**	4.556	0.008	**0.000**

Table 4. Firm's loss functions for $VaR^{l}_{t|t+1}(\alpha)$ and $VaR^{s}_{t|t+1}(\alpha)$ with c = 0.000114 (average WIBOR O/N rate)

α	Portfolio with a = (1, ..., 1)′			Portfolio with $\omega_{\tau i}$ = 1/50		
	n-variate MSF-SBEKK	univariate MSF-SBEKK	CAViaR	*n*-variate MSF-SBEKK	univariate MSF-SBEKK	CAViaR
	Long trading position					
0.01	133.287	50.500	**20.481**	55.266	5.557	**4.925**
0.025	221.942	101.545	**70.783**	124.009	43.677	**42.551**
0.05	350.354	193.563	**186.698**	229.488	108.789	**103.268**
0.1	569.812	**372.356**	384.746	405.253	**234.217**	316.575
	Short trading position					
0.1	536.244	257.159	**155.504**	286.677	117.510	**74.371**
0.05	289.598	89.717	**53.730**	114.419	34.718	**14.099**
0.025	144.763	24.849	**16.616**	35.775	9.362	**0.038**
0.01	44.113	1.773	**1.556**	4.545	0.023	**0.021**

Figure 3. $VaR_{t|t+1}$ (0.05) and $ES_{t|t+1}$(0.05) results based on the univariate MSF-SBEKK model, $\omega_{\tau i}$ = 1/50

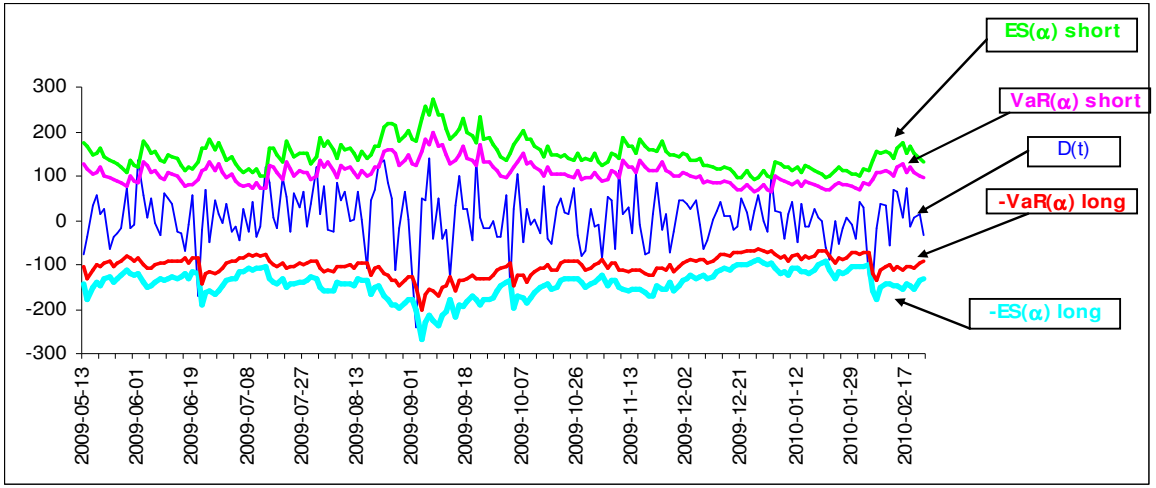


Figure 4. $VaR_{t|t+1}$ (0.05) and $ES_{t|t+1}$(0.05) results based on the *n*-variate MSF-SBEKK model, $\omega_{\tau i}$ = 1/50

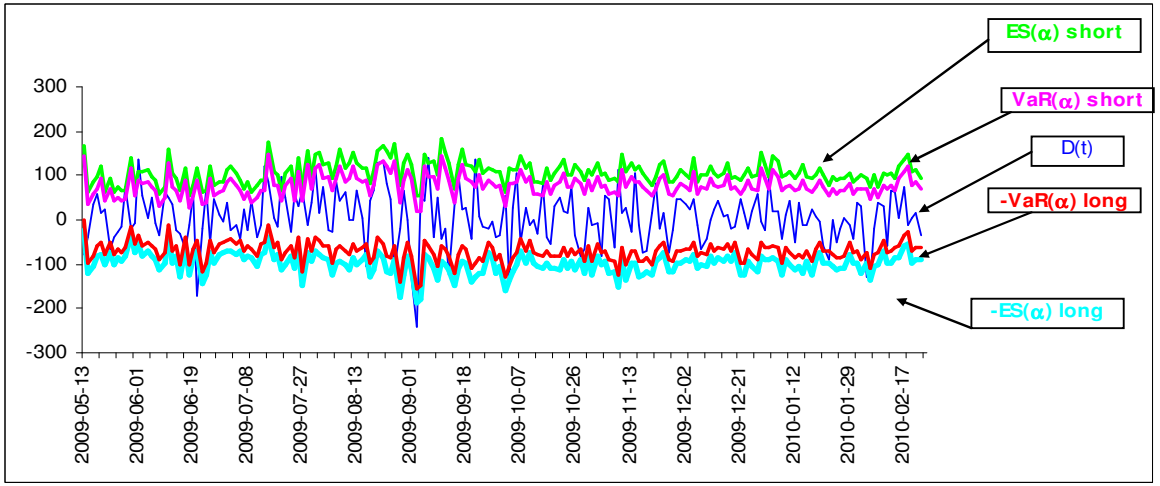


Figure 5. $VaR_{t|t+1}$ (0.05) and $ES_{t|t+1}$(0.05) results based on the CAViaR/CARE model, $\omega_{\tau i}$ = 1/50

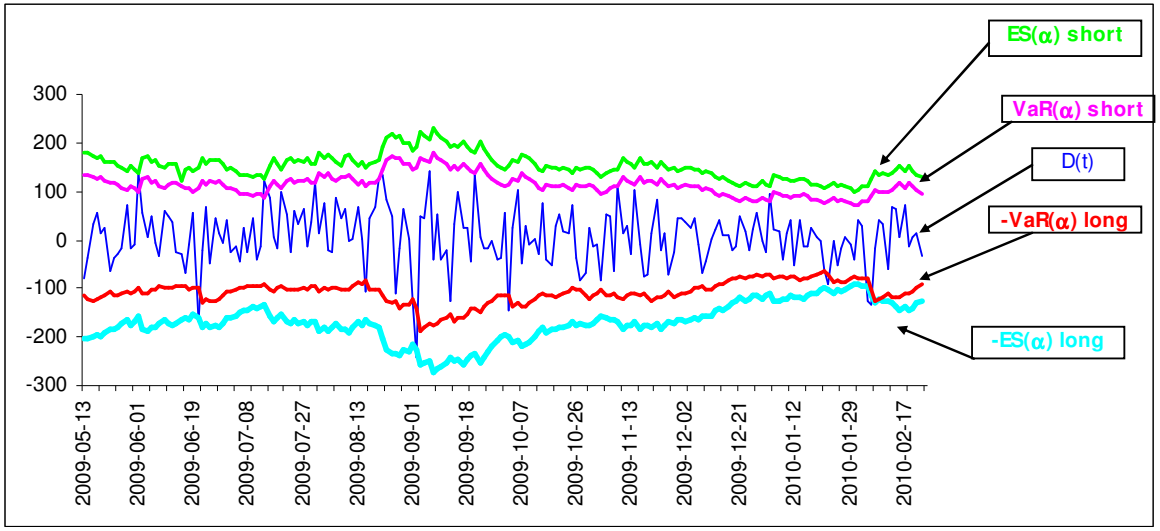


Figure 6. $VaR_{t/t+1}$ (0.05) results, $\omega_{\tau i} = 1/50$

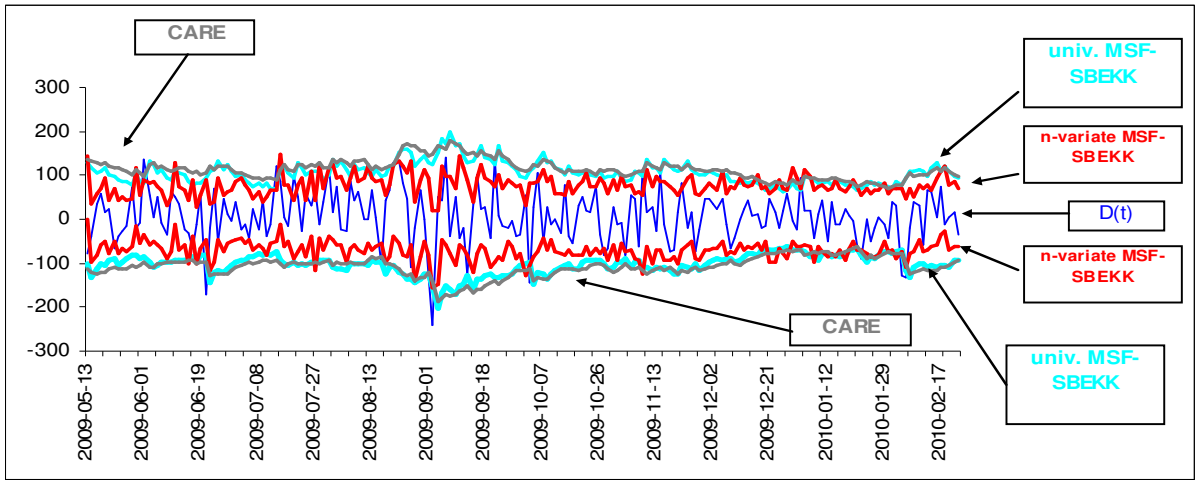


Figure 7. $ES_{t|t+1}$(0.05) results, $\omega_{\tau i} = 1/50$

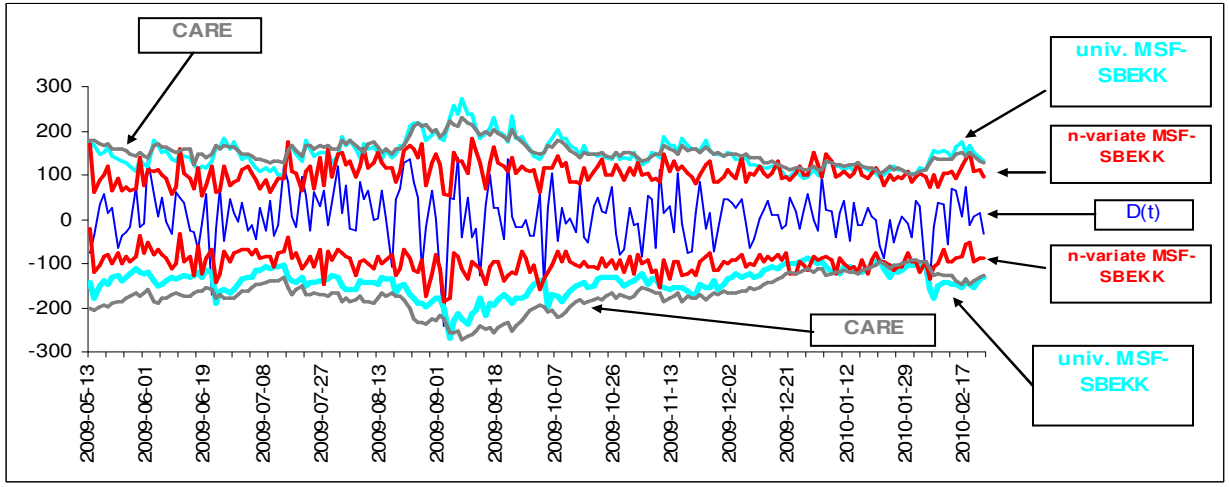


Table 5. Correlation coefficients between $VaR^{l}_{t|t+1}(\alpha)$ for $\alpha = 0.01$ and $\alpha = 0.05$ (upper part), for $\alpha = 0.025$ and $\alpha = 0.1$ (lower part), $\omega_{ti} = 1/50$

α=0.01 / α=0.025	*n*-variate MSF-SBEKK	univariate MSF-SBEKK	CAViaR	α=0.05 / α=0.1	*n*-variate MSF-SBEKK	univariate MSF-SBEKK	CAViaR
n-variate MSF-SBEKK	1	0.346	0.216	*n*-variate MSF-SBEKK	1	0.311	0.200
univariate MSF-SBEKK	0.325	1	**0.912**	univariate MSF-SBEKK	0.317	1	**0.916**
CAViaR	0.204	**0.944**	1	CAViaR	0.197	**0.757**	1

Table 6. Correlation coefficients between $ES^{I}_{t|t+1}(\alpha)$ for $\alpha = 0.01$ and $\alpha = 0.05$ (upper part), for $\alpha = 0.025$ and $\alpha = 0.1$ (lower part), $\omega_{ti} = 1/50$

α=0.01 / α=0.025	*n*-variate MSF-SBEKK	univariate MSF-SBEKK	CAViaR/ CARE	α=0.05 / α=0.1	*n*-variate MSF-SBEKK	univariate MSF-SBEKK	CAViaR/ CARE
n-variate MSF-SBEKK	1	0.363	0.174	*n*-variate MSF-SBEKK	1	0.331	0.197
univariate MSF-SBEKK	0.346	1	**0.776**	univariate MSF-SBEKK	0.316	1	**0.852**
CAViaR/ CARE	0.161	**0.787**	1	CAViaR/ CARE	0.148	**0.854**	1

Measure of predictive out-of-simple performance for ES

Zhu and Galbraith [2009]:

1) Mean error:

$$ME_s(\alpha) = ES_s^A(\alpha) - AL_s(\alpha)$$

where

- $ES_s^A(\alpha)$ is the average predictive ES:

$$ES_s^A(\alpha) = \frac{1}{J_s} \sum_{t=T}^{T+p-1} I_{(-\infty,0)}(D_{t|t+s} + VaR_{t|t+s}^l(\alpha)) ES_{t|t+s}^l(\alpha),$$

- $AL_s(\alpha)$ is the average loss on the portfolio when the loss is larger than $VaR_{t/t+s}(\alpha)$:

$$AL_s(\alpha) = \frac{1}{J_s} \sum_{t=T}^{T+p-1} I_{(-\infty,0)}(D_{t|t+s} + VaR_{t|t+s}^l(\alpha)) \mid D_{t|t+s} \mid,$$

2) Mean absolute error:

$$MAE_s(\alpha) = \frac{1}{J_s} \sum_{t=T}^{T+p-1} I_{(-\infty,0)}(D_{t|t+s} + VaR^l_{t|t+1}(\alpha)) \, |ES^l_{t|t+s}(\alpha) - AL_s(\alpha)|,$$

where $J_s = \sum_{t=T}^{T+p-1} I_{(-\infty,0)}(D_{t|t+s} + VaR^l_{t|t+s}(\alpha))$

Table 7. Mean error for $ES^{l}_{t|t+1}(\alpha)$ or $ES^{s}_{t|t+1}(\alpha)$

α	Portfolio with a = (1, ..., 1)′			Portfolio with $\omega_{\tau i}$ = 1/50		
	n-variate MSF-SBEKK	univariate MSF-SBEKK	CAViaR/ CARE	*n*-variate MSF-SBEKK	univariate MSF-SBEKK	CAViaR/ CARE
	Long trading position					
0.01	**-15.644**	-23.710	21.777	**-12.354**	23.569	85.407
0.025	-9.576	-3.392	**1.326**	-8.339	**-4.611**	42.660
0.05	-7.436	-4.584	**3.964**	-12.410	**-0.298**	16.658
0.1	-14.988	-6.500	**-0.154**	-13.557	**0.086**	33.693
	Short trading position					
0.1	-14.587	-1.331	**0.236**	**-2.567**	12.798	19.319
0.05	-15.284	7.228	**7.016**	**3.285**	15.357	22.119
0.025	**-7.102**	15.467	14.581	9.929	**9.305**	30.563
0.01	**4.731**	24.866	14.279	**14.633**	30.913	--

Table 8. Mean absolute error *for* $ES^{l}_{t|t+1}(\alpha)$ or $ES^{s}_{t|t+1}(\alpha)$

α	Portfolio with a = (1, ..., 1)′			Portfolio with $\omega_{\tau i}$ = 1/50		
	n-variate MSF-SBEKK	univariate MSF-SBEKK	CAViaR/ CARE	*n*-variate MSF-SBEKK	univariate MSF-SBEKK	CAViaR/ CARE
	Long trading position					
0.01	40.867	23.710	**21.777**	49.082	**48.459**	105.015
0.025	31.304	44.111	**19.748**	33.028	**38.637**	78.232
0.05	24.329	29.538	**20.304**	30.436	**27.452**	46.127
0.1	24.497	24.592	**17.976**	26.162	**21.287**	35.339
	Short trading position					
0.1	24.996	**15.057**	16.162	20.126	24.606	**21.752**
0.05	24.159	21.190	**18.754**	26.104	25.275	**24.774**
0.025	20.713	23.611	**23.050**	20.237	**11.374**	30.563
0.01	22.391	45.440	**14.279**	20.800	30.913	--

Table 9. $VaR_{t|t+s}$ (0.05) results based on the **univariate** MSF-SBEKK model, $\omega_{\tau i}$ = 1/50

s	1	2	3	4	5	6	7	8	9	10
Long trading position										
FR	0.04	0.02	0.035	0.025	0.03	0.03	0.01	0.005	0.005	0
p-value for the Kupiec test	0.502	0.028	0.305	0.074	0.162	0.162	0.002	0.000	0.000	--
Lopez loss fun.	**108.82**	**303.04**	**240.16**	**71.29**	**20.42**	**36.11**	**2.03**	**3.35**	0.05	0
Short trading position										
FR	0.04	0.035	0.05	0.04	0.04	0.04	0.04	0.055	0.045	0.055
p-value for the Kupiec test	0.502	0.305	1.000	0.502	0.502	0.502	0.502	0.749	0.742	0.749
Lopez loss fun.	**34.750**	**106.55**	**172.80**	**291.73**	**330.46**	**305.43**	132.96	145.19	258.43	247.49

Table 10. $VaR_{t|t+s}$ (0.05) results based on the ***n*-variate** MSF-SBEKK model, $\omega_{\tau i}$ = 1/50

s	1	2	3	4	5	6	7	8	9	10
Long trading position										
FR	0.090	0.065	0.055	0.050	0.040	0.025	0.020	0.010	0.000	0.000
p-value for the Kupiec test	0.019	0.351	0.749	1.000	0.502	0.074	0.028	0.002	--	--
Lopez loss fun.	229.57	537.65	512.57	218.78	47.73	214.22	125.80	18.02	0.00	0.00
Short trading position										
FR	0.120	0.085	0.070	0.065	0.050	0.060	0.050	0.065	0.055	0.045
p-value for the Kupiec test	0.000	0.038	0.220	0.351	1.000	0.529	1.000	0.351	0.749	0.742
Lopez loss fun.	114.53	176.29	367.92	474.20	583.03	450.83	**116.23**	**115.80**	**164.29**	**162.26**

Figure 8. Lopez loss function values for $VAR^{l}_{t|t+s}$ (0.05), ω_{ti} = 1/50, s =1, 2, ..., 10

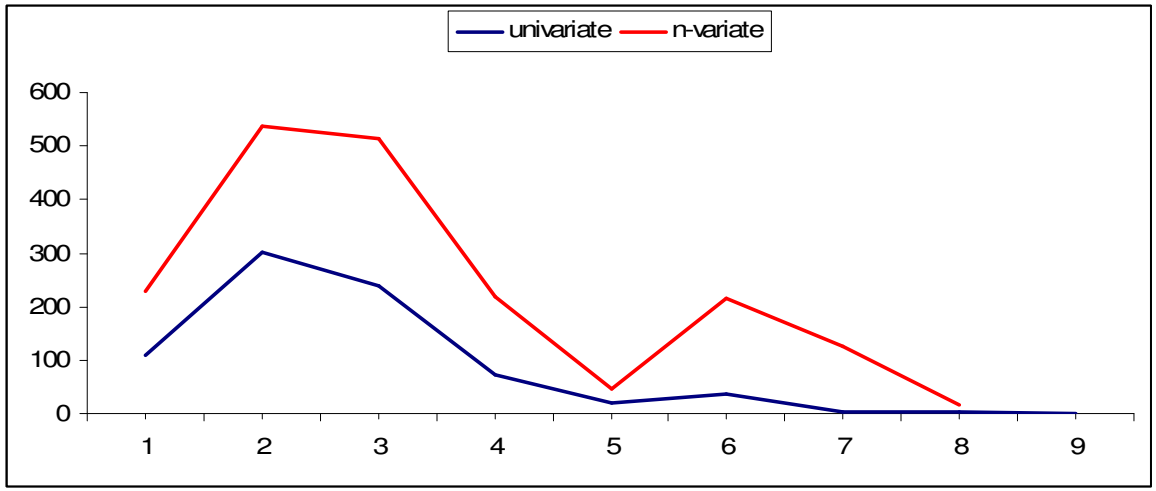


Figure 9. Lopez loss function values for $VAR^{s}_{t|t+s}$ (0.05), ω_{ti} = 1/50, s =1, 2, ..., 10

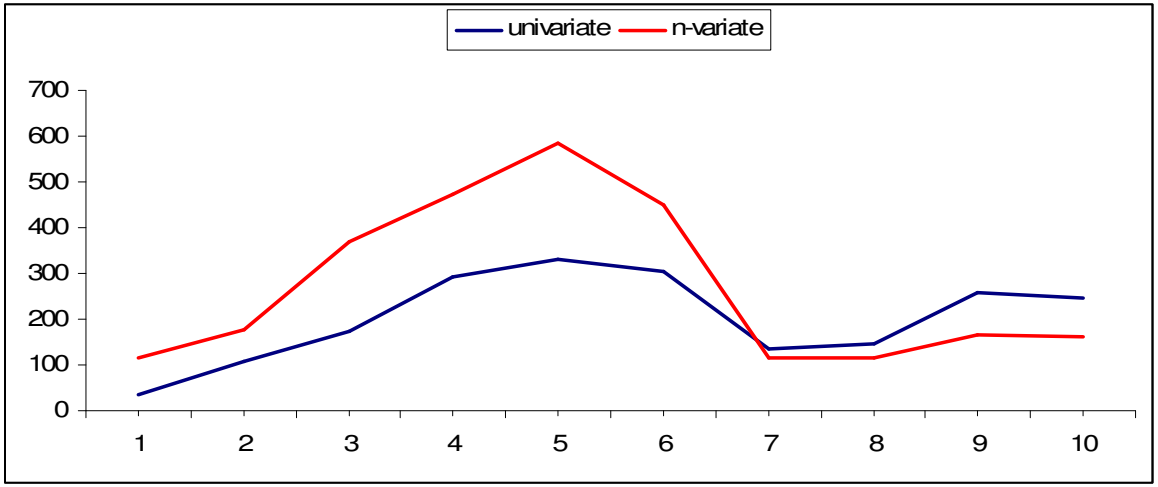


Table 11. $ES^{l}_{t|t+s}$ (0.05) results based on the **univariate** MSF-SBEKK model, $\omega_{\tau i}$ = 1/50

s	1	2	3	4	5	6	7	8	9	10
FR	0.040	0.020	0.035	0.025	0.030	0.030	0.010	0.005	0.005	0.000
AL	142.738	242.405	240.867	240.253	214.166	271.849	242.791	266.853	269.926	--
ES_A	142.440	184.580	241.170	262.578	260.841	328.175	315.474	328.513	363.212	--
ME	**-0.298**	-57.825	**0.303**	22.325	**46.675**	56.326	72.683	**61.659**	93.286	--
MAE	285.178	426.985	482.037	502.831	**475.007**	600.024	**558.265**	**595.366**	633.138	--

Table 12. $ES^{l}_{t|t+s}$ (0.05) results based on the ***n*-variate** MSF-SBEKK model, $\omega_{\tau i}$ = 1/50

s	1	2	3	4	5	6	7	8	9	10
FR	0.090	0.065	0.055	0.050	0.040	0.025	0.020	0.010	0.000	0.000
AL	103.958	159.685	200.719	228.810	241.120	287.136	323.616	338.440	--	--
ES_A	91.548	140.876	189.018	236.260	288.469	306.340	342.510	400.207	--	--
ME	-12.410	**-18.808**	-11.701	**7.451**	47.349	**19.205**	**18.893**	61.767	--	--
MAE	**195.505**	**300.561**	**389.738**	**465.070**	529.588	**593.476**	666.126	738.646	--	--

Table 13. $ES^{s}_{t|t+s}$ (0.05) results based on the **univariate** MSF-SBEKK model, $\omega_{\tau i}$ = 1/50

s	1	2	3	4	5	6	7	8	9	10
FR	0.040	0.035	0.050	0.040	0.040	0.040	0.040	0.055	0.045	0.055
AL	118.021	182.901	223.742	281.317	299.185	331.351	336.279	340.296	399.622	388.952
ES_A	133.377	196.498	252.128	300.115	333.295	384.346	406.770	428.851	472.582	497.844
ME	15.357	13.597	28.386	18.798	34.111	52.994	70.490	88.555	72.960	108.892
MAE	**25.275**	34.787	46.428	49.994	48.387	66.538	72.681	88.555	76.164	108.892

Table 14. $ES^{s}_{t|t+s}$ (0.05) results based on the ***n*-variate** MSF-SBEKK model, $\omega_{\tau i}$ = 1/50

s	1	2	3	4	5	6	7	8	9	10
FR	0.120	0.085	0.070	0.065	0.050	0.060	0.050	0.065	0.055	0.045
AL	92.424	154.958	213.392	246.650	279.601	306.237	327.124	340.355	384.463	416.780
ES_A	95.709	163.295	215.283	254.120	280.114	323.938	377.204	403.919	441.862	478.828
ME	**3.285**	**8.337**	**1.891**	**7.470**	**0.513**	**17.701**	**50.080**	**63.564**	**57.399**	**62.048**
MAE	26.104	**34.498**	**34.932**	**27.775**	**21.416**	**30.948**	**50.199**	**63.913**	**57.399**	**62.048**

Figure 10. Mean absolute errors for $ES^{l}_{t|t+s}$ (0.05), $\omega_{\tau i}$ = 1/50, s =1, 2, ..., 10

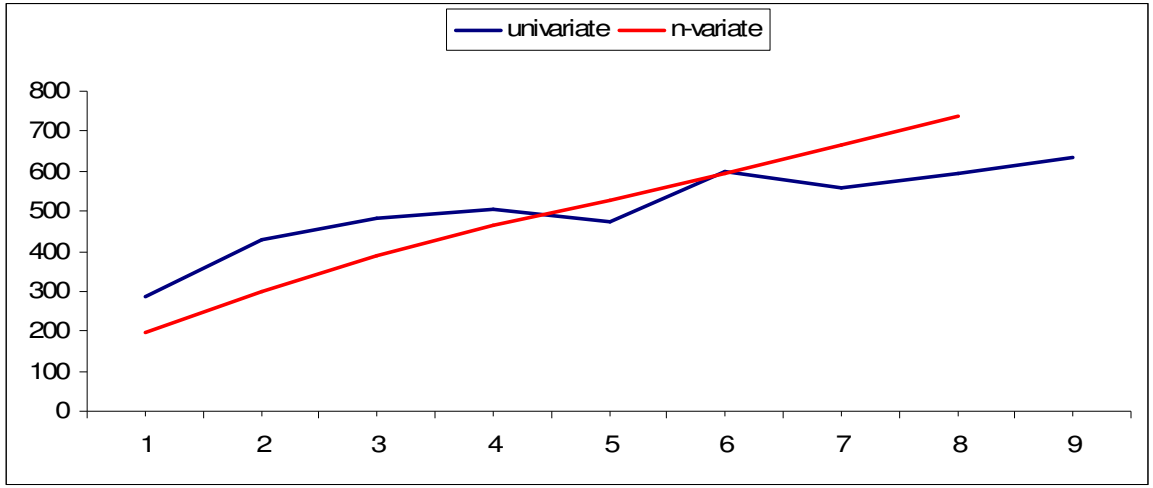
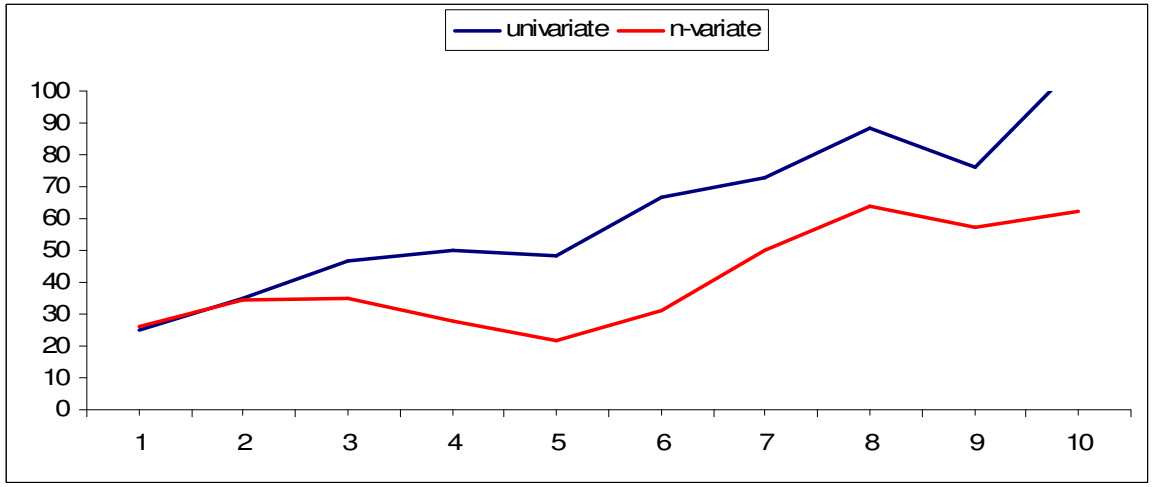


Figure 11. Mean absolute errors for $ES^{s}_{t|t+s}$ (0.05), $\omega_{\tau i}$ = 1/50, s =1, 2, ..., 10



7. Concluding remarks

- the Bayesian approach and MSF-SBEKK model have been successfully used to build the predictive distribution of the future value of large portfolios (taking into account their full conditional covariance structure: individual volatilities and correlations)
- for VaR assessment, univariate modelling (of portfolio value – instead of portfolio components) is enough.

While the *n*-variate MSF–BEKK model may occur practical and useful in optimisation problems for large portfolios, its role in VaR analysis is doubtful.

- univariate special case of the MSF–SBEKK model behaves very well and successfully competes with the CAViaR and CARE non-parametric specifications
- the Bayesian approach to VaR and ES analysis is fully relevant and practical. Conditioning on observed data as well as inference on non-linear functions of unobserved quantities (future logarithmic returns) are necessary for any appropriate VaR analysis. Both are natural and easy within Bayesian statistics, equipped with the Markov Chain Monte Carlo simulation tools

Thank you for your attention!

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