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Behaviour of Exchange Rates and Returns:

Long Memory and Cointegration

Or:

Quite a few methods of financial econometrics come from physics/technical sciences

We study behaviour of time series of daily closing values for USD and EURO vs. PLN exchange rates and their returns.

Financial data series are characterized by changing volatility, excess kurtosis and asymmetry of their probability density, and show signs of long memory.

Due to volatility clustering, can be modelled with use of ARCH and GARCH type models, in which first equation -- describing expected value of the series -- can be of ARMA type, second equation describes conditional variance.

It has been shown that additional explanatory variables in the expected value equation can improve quality of modelling and of forecasts. For bilateral exchange rates models such additional variables can be stock indices of corresponding countries.

Cointegration analysis of exchange rates and stock indices is performed to check whether there is a stable dynamic economic equlibrium between them.

The tools applied in such research stem originally from technical if not physical applications: the Hurst exponent, from hydrological study of 1950's; the ARMA models, from Box and Jenkins fundamental monograph of 1970's; cointegration analysis, from Engle and Granger concepts of equlibrium path as stable attractor; the Hansen stability tests and time-varying-parameter cointegration analysis uses methods of spectral analysis to estimate in semiparametric way long-run variance matrix.

Operational definition of **stationarity**:

1) expected value E[X_t] constant, independent of time;

2) variance $D^2(X_t)$ constant and finite, independent of time;

3) covariance Cov(X_t, X_s) depends only on | $t - s$ |.

Stationary process is characterized by:

1) Its mean: $\bar{x} = \frac{1}{n}\sum_{t=1}^{n} x_t$

2) Covariance:

$$C_\tau = \frac{1}{n-\tau}\sum_{t=1}^{n-\tau}(x_t - \bar{x})(x_{t-\tau} - \bar{x})$$

3) Autocorrelation function:

$$R_\tau = \frac{C_\tau}{C_0} = \hat{\rho}_\tau$$

4) Periodogram:

$$I(\omega) = \frac{1}{2\pi}\sum_{\tau=-n}^{n} C_\tau \cos\omega\tau$$

5) Spectral density function:

$$f(\omega_j) = \frac{1}{\pi}\sum_{\tau=0}^{m} \lambda_{m,\tau} C_\tau \cos\omega_j\tau$$

Testing for nonstationarity:

1) **Dickey-Fuller test**: H_0: y_t is nonstationary, vs. H_1: y_t is stationary:
Based on regression:

$$\Delta y_t = \mu + \delta y_{t-1} + \sum_{j=1}^{k} \gamma_j \Delta y_{t-j} + \varepsilon_t$$

Parameter $\delta = \rho - 1$ corresponds to an autoregression parameter: if $\delta = 0$, then $\rho = 1$ and we have a random walk, if $\delta < 0$, then $\rho < 1$ and the process is stationary.

Test statistics: ADF= $\hat{\delta}/s_{\hat{\delta}}$, to be compared with a proper critical value.

2) **Kwiatkowski, Phillips, Schmidt and Shin** (KPSS) test[1]:
H_0: y_t is stationary, vs. H_1: y_t is nonstationary:
Based on representation:

$$y_t = r_t + \xi + \varepsilon_t$$
$$r_t = r_{t-1} + u_t$$

Parameter: variance of u_t, σ_u^2: if it is 0, then r_t=const. and y_t is stationary, if $\sigma_u^2 > 0$, r_t is a random walk and y_t is nonstationary.
(The test statistics has a very complex distribution, asymptotic critical values given by KPSS are computed with use of limits which are quite complicated functions of Brownian bridges).

[1] D. Kwiatkowski, P. C. B. Phillips, P. Schmidt, and Y. Shin (1992): "Testing the Null Hypothesis of Stationarity against the Alternative of a Unit Root" Journal of Econometrics 54, 159–178.

Exchange rates, stock indices: behaviour similar to that of a **random walk process:**

$$y_t = y_{t-1} + \varepsilon_t$$

If $y_0 = 0$, then $y_t = \sum_{i=1}^{t} \varepsilon_i \rightarrow$ is a long-memory process;

$$D^2(y_t) = \sum_{i=1}^{t} D^2(\varepsilon_i) = t\sigma^2 \rightarrow \text{is nonstationary.}$$

But $\Delta y_t = \varepsilon_t$ is stationary, hence y_t is an example of integrated process.

The process is **integrated**, with order of integration *d*, if it is nonstationary but its difference is stationary[2]. Order of integration is the least *integer* number of differences sufficient for obtaining stationarity.

$$y_t \sim I(1): y_t \ nonstationary, \Delta y_t \ stationary.$$

Cointegration of series $y_t, x_{1t}, x_{2t}, \ldots, x_{kt} \sim I(1)$ means that the series are nonstationary, but there is a linear combination which is stationary.
More general case: linear combination with lower order of integration than the variables (see Engle and Granger (1987)).

Vector of coefficients of this linear combination is called *cointegrating vector*.

Interpretation (see Maddala and Kim)[3]:

If we know that there is a stable long-run economic equilibrium, *then* – as series of question are as a rule nonstationary – *one of possible cointegration vectors corresponds to this economic relationship.*

Interpretation (C.W.J. Granger): cointegrating relationship is an attractor.

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[2] Robert F. Engle and Clive W.J. Granger, "Co-integration and Error Correction: Representation, Estimation, and Testing", *Econometrica*, 1987, 55(2), 251-76.

Sir **Clive William John Granger** (September 4, 1934 – May 27, 2009) was a British economist, and Professor Emeritus at the University of California, San Diego. In 2003, Granger was awarded theNobel Memorial Prize in Economic Sciences. In bestowing this honor, the Royal Swedish Academy of Sciences committee recognized that Granger and Robert F. Engle had made fundamental discoveries in the analysis of time series data and that this work was widely known fundamentally to have changed the way economists analyze financial and macroeconomic data.

[3] Maddala, G.S. and In-Moo Kim, "Unit roots, cointegration and structural change", Cambridge University Press, 1998.

Fig. 1: Closing daily values of USDPLN exchange rate

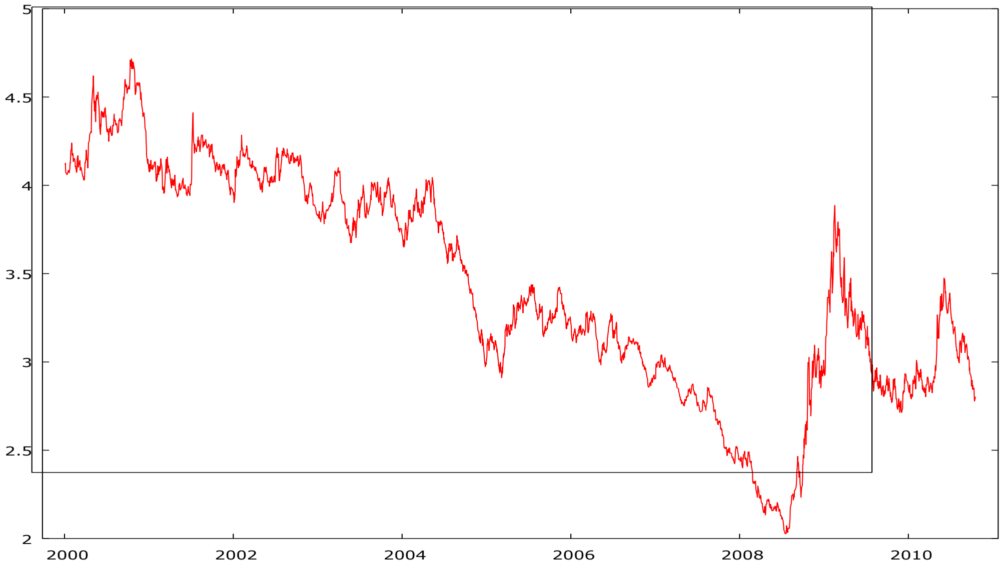
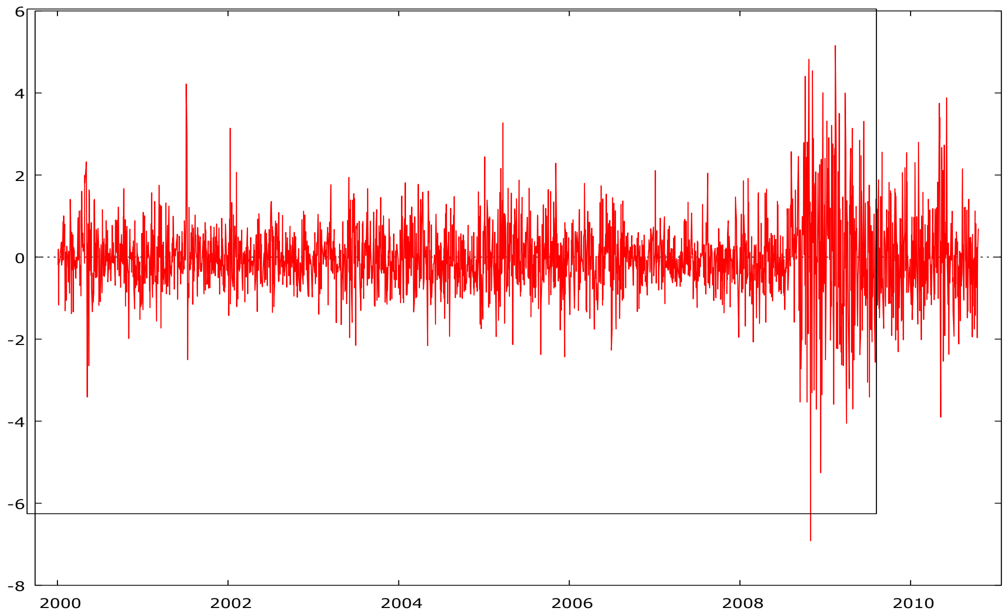


Fig. 2. Logarithmic returns of USDPLN exchange rate daily data.



Typical features of financial time series:

- Higher volatility – higher risk of investment;
- Asymmetric, non-normal distribution;
- Volatility clustering – i.e., serial correlation of conditional variance.

Fractional integration parameter:

A real number *d*, such that for a nonstationary series $\{y_t\}$ increments are stationary:

$$\Delta^d y_t = \varepsilon_t,$$

where $\Delta^d y_t$ are defined as:

$$\Delta^d = (1-L)^d = \sum_{k=0}^{\infty} \binom{d}{k} (-1)^k L^k = \sum_{k=0}^{\infty} \frac{\Gamma(k-d)}{\Gamma(-d)\Gamma(k+1)} L^k,$$

and *L* denotes lag parameter.

Properties of a series depend on d :

- If d =1, the process is integrated and has infinite variance,
- If d >1, the process is also nonstationary, and effects of external shocks increase in time.
- If 0.5 $\leq d < 1$, variance is infinite, hence the process is also nonstatonary, but in long time is mean-reverting (see Hosking (1981)[4]). Effects of shocks last for a long time.
- If 0 < d < 0.5, the process is stationary, mean-reverting, with finite variance.
- If d = 0, the process is mean-reverting in a short time, has finite variance, and shock effect diminish quickly.
- If d <0, the process is antipersistent and stationary.

Applications of fractional integration to financial data: exchange rates, asset returns, interest rates, inflation – see Baillie (1996)[5]

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[4] Hosking, J.R.M. (1981), "Fractional differencing", Biometrika, 68(1), 165–176.

[5] Baillie, Richard T., (1996) "Long memory processes and fractional integration in econometrics," Journal of Econometrics, 73(1), 5-59.

Estimation of fractional integration parameter:

The Geweke and Porter-Hudak[6] method, among others, is based on periodogram regression:

For a stationary series X and white noise u , if $\Delta^d X_t = u_t$ and u_t is stationary with zero mean and continuous spectral density, $f_u(\omega) > 0$, then:

$$f_X(\omega) = |1 - \exp(i\omega)|^{-2d} \ f_u(\omega)$$

P.C.B. Phillips (1999) shows that[7] for a nonstationary series this is a limit of periodogram ordinates. For fundamental frequencies $\omega_s = 2\pi s/N$, where N is number of observations, $s = 1,2,\ldots,m$, a regression

$$\log I_x(\omega_s) = c - d \log|1 - \exp(i\omega_s)| + residual$$

is estimated with OLS, hence this kind of fractional integration parameter estimates are called *periodogram regressions*.

See also Phillips (1999) for corrections improving accuracy of periodogram ordinates computations.

In periodogram regression, we can test hypothesis about d, e.g., whether d =0 for a stationary series, or d=1 for a nonstationary series.

(Another method of fractional integration parameter estimation was introduced by Whittle.)

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[6] Geweke, John i Porter-Hudak, Susan (1983), The estimation and application of long-memory time series models, Journal of Time Series Analysis, 4, 221–228.

[7] P.C.B. Phillips, "Unit root log periodogram regression", Cowles Foundation Discussion Paper No. 1244, http://cowles.econ.yale.edu/P/cd/d12a/d1244.pdf

The Hurst exponent etc.

See e.g., Edgar Peters “Chaos and order in capital markets”, Wiley, 1996 (2nd edition), “Fractal market analysis: Applying chaos theory to investment and economics”, Wiley, 1994. „Modelowanie procesów na rynku kapitałowym za pomocą multifraktali” (Capital market modelling with use of multifractals), Adrianna Mastalerz-Kodzis ; Katowice, University of Economics in Katowice, 2003

Harold Edwin Hurst (1880–1978) – introduced the Hurst exponent to optimize size of the Nile dam[8].

The Hurst exponent is used as **a measure of long memory of a series**:

0<H<0.5 indicates a series with negative autocorrelation,
H> 0.5 indicates a series with positive autocorrelation,
H=0.5 indicates a random walk.

Estimation: Let r_t denote logarithmic returns, $m(N,t_0)=\sum_{t=t_0+1}^{t_0+N} r_t / N$ – mean of a series,

$$S(N,t_0)=\left\{\frac{1}{N}\sum_{t=t_0+1}^{t_0+N}\left(r_t-m(N,t_0)\right)^2\right\}^{1/2}$$ – biased estimator of standard deviation

Partial sums and range of partial sums of deviations from a mean are defined as

$$X(N,t_0,\tau)\equiv\sum_{t=t_0+1}^{t_0+\tau}(r_t-m(N,t_0)) \text{ for } 1\le\tau\le N, \qquad \text{(R1)}$$

$$R(N,t_0)\equiv\max_{\tau} X(N,t_0,\tau)-\min_{\tau} X(N,t_0,\tau). \qquad \text{(R2)}$$

Rescaled range statistics, defined as $[R/S](N)\equiv\dfrac{\sum_{t_0} R(N,t_0)}{\sum_{t_0} S(N,t_0)}$.

is equal to $[R/S](N)\approx(aN)^H$, where a – constant term, H – the Hurst exponent.

8 Hurst, H. (1951), *Long term storage capacity of reservoirs*, Transactions of the American Society of Civil Engineers, 116, 770–799.

Hence $\log[\widehat{R/S}]\,(N_i) = \hat{c} + \widehat{\mathrm{H}}\log N_i$

Example: Rescaled range figures for logarithmic returns of USDPLN daily data:
(logs are to base 2)

```
 Size      RS(avg)    log(Size)      log(RS)
 2641       72.964       11.367       6.1891
 1320       52.398       10.366       5.7114
  660       32.131       9.3663       5.0059
  330       23.909       8.3663       4.5795
  165       16.679       7.3663       4.0600
   82       11.353       6.3576       3.5051
   41       7.2341       5.3576       2.8548
   20       4.7010       4.3219       2.2330
   10       3.0911       3.3219       1.6281

Regression results (n = 9)

                    coeff   std. error
   Intercept     -0.17730     0.072459
       Slope      0.56447    0.0092902

Estimated Hurst exponent = 0.564465
```

1. **The augmented Dickey-Fuller test:**
   H0: nonstationarity of a series vs. H1: stationarity

The ADF test for:	Sample up to 30th April		Sample up to 19th November 2010	
	Variable	Logarithmic returns	Variable	Logarithmic returns
SP500close	-1.82469 [0.369]	-12.2809 [0.000]	-1.90043 [0.332]	-12.5817 [0.000]
WIG20close:	-1.14794 [0.699]	-21.0512 [0.000]	-1.15186 [0.697]	-21.6656 [0.000]
USDPLNclose	-1.56590 [0.500]	-9.89291 [0.000]	-1.61695 [0.474]	-9.84206 [0.000]
EURUSDclose	-1.26116 [0.650]	-10.6862 [0.000]	-1.28047 [0.641]	-10.6264 [0.000]
EURPLNclose	-2.12460 [0.235]	-9.18958 [0.000]	-2.20862 [0.203]	-9.45797 [0.000]

2. Fractional integration parameter estimates:

Estimates for a series:	Geweke and Porter-Hudak estimates: *t-Statistics (for H0: d=1) in parentheses*	Estimates for log returns of a series:	Geweke and Porter-Hudak estimates: *t-Statistics (for H0: d=0) in parentheses*
SP500close	0.993118 (0.0527) [0.1306]	SP500close	0.000702313 (0.0550)
WIG20 close	1.09943 (0.0785) [1.2660]	WIG20 close	0.0780061 (0.063505)
USDPLNclose	1.07213 (0.0610703) ; [1.165]	USDPLNclose	**0.106393 (0.0575171) [1.850]**

3. The Hurst exponents:

Hurst exponent:	for a variable	for logarithmic returns
SP500close	0.968876	0.551037
WIG20 close	1.00452	0.568485
USDPLNclose	1.00179	0.564514

Values of the Hurst exponent estimates for log returns are close to value for a random walk; see also Czekaj, Woś, Żarnowski (2001)[9] for test on *H.*

[9] Czekaj, M. Woś, J. Żamowski, Efektywność giełdowego rynku akcji w Polsce z perspektywy dziesięciolecia, PWN, Warszawa 2001

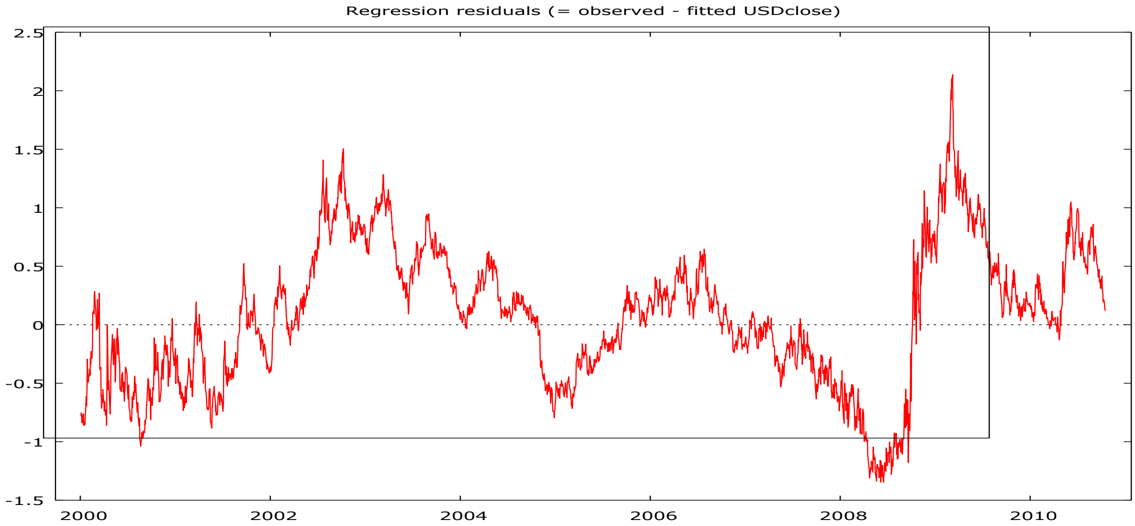
4. Cointegration analysis for exchange rate and indices:

```
Model: OLS, using observations 2000/01/04-2010/10/18 (T =
2641)
Dependent variable: USDclose
                  coefficient    std. error     t-ratio    p-value
----------------------------------------------------------------
SP500close     0.00509112    3.82529e-05    133.1      0.0000   ***
WIG20close    -0.00125702    2.04321e-05    -61.52     0.0000   ***

Mean dependent var    3.432219    S.D. dependent var    0.623092
Sum squared resid     911.3855    S.E. of regression    0.587667
R-squared             0.971640    Adjusted R-squared    0.971629
F(2, 2639)            45207.26    P-value(F)            0.000000
Log-likelihood       -2342.475    Akaike criterion      4688.949
Schwarz criterion     4700.707    Hannan-Quinn          4693.206
rho                   0.988944    Durbin-Watson         0.021511
```

Stock indices use as explanatory variables for an exchange rate was suggested by a paper by Bauwens, Rime and Succarat on daily returns of Norwegian krona[10].

If (1, –0,00509, 0.00126) is a cointegrating vector for USDPLN, SP500 and WIG20, then residuals of this regression should be stationary – do not quite seem to be:



And the ADF test statistics for residuals, –2.81252, has asymptotic p-value 0.056 –higher than 5%.

[10] Bauwens, L., Pohlmeier, W., Veredas, D. (2008) *High Frequency Financial Econometrics. Recent Developments*, Physica-Verlag A Springer Company, Heidelberg.
Bauwens, L., Rime, D., Succarat, G. (2008) *Exchange Rate Volatility and the Mixture of Distribution Hypothesis*, [in:] Bauwens et al., 7–29.

5. ARIMA model for USDPLN:

We estimate an ARIMA model for exchange rate, using first difference, and with corresponding stock indices as additional explanatory variables:

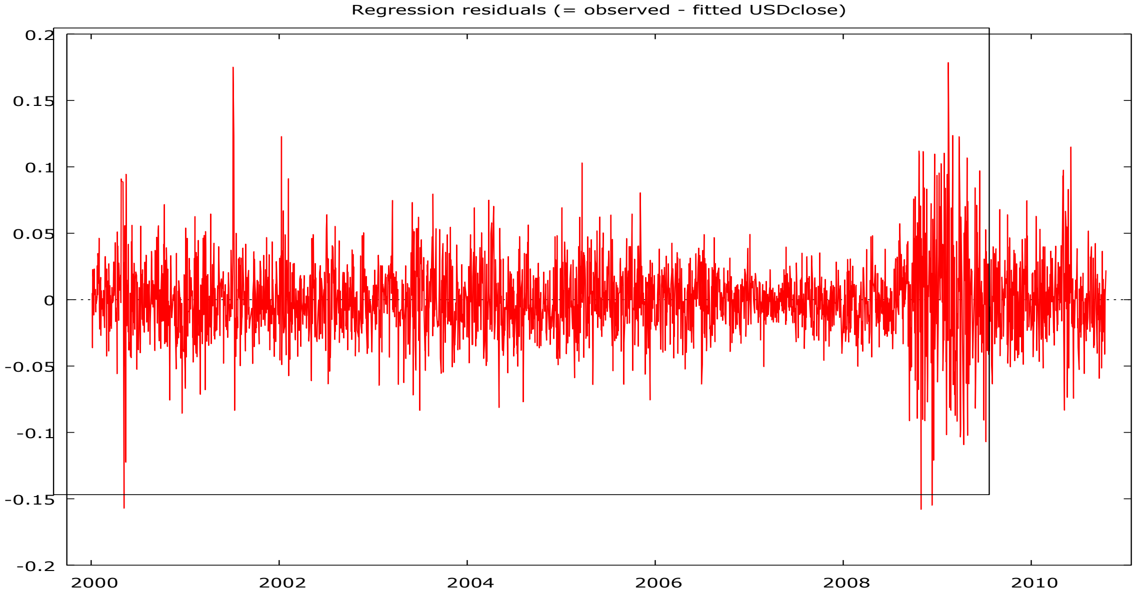
```
Model:  ARMAX, using observations 2000/01/05-2010/10/18 (T = 2640)
Estimated using Kalman filter (exact ML)
Dependent variable: (1-L) USDclose
Standard errors based on Hessian

                coefficient    std. error       z         p-value
  -------------------------------------------------------------
  const         -0.000448115   0.000529548    -0.8462    0.3974
  phi_1          1.30899       0.216640        6.042     1.52e-09   ***
  phi_2         -0.559846      0.159075       -3.519     0.0004     ***
  theta_1       -1.27272       0.226735       -5.613     1.99e-08   ***
  theta_2        0.505183      0.168287        3.002     0.0027     ***
  SP500close    -0.000178384   3.87039e-05    -4.609     4.05e-06   ***
  WIG20close    -0.000190832   1.58094e-05   -12.07      1.51e-033  ***

Mean dependent var   -0.000498   S.D. dependent var    0.030611
Mean of innovations   2.83e-06   S.D. of innovations   0.029356
Log-likelihood        5568.628   Akaike criterion    -11121.26
Schwarz criterion   -11074.23   Hannan-Quinn        -11104.23

                        Real   Imaginary     Modulus   Frequency
  -----------------------------------------------------------
  AR
    Root  1           1.1691     -0.6477      1.3365     -0.0805
    Root  2           1.1691      0.6477      1.3365      0.0805
  MA
    Root  1           1.2597     -0.6267      1.4069     -0.0735
    Root  2           1.2597      0.6267      1.4069      0.0735
  -----------------------------------------------------------
```

All variables significant, roots of polynomials have moduli greater than 1, hence the model is stable.



6. **The Engle[11] test of the ARCH effect**:

The Engle test of the ARCH effect is based on the regression

$$e_t^2 = \alpha_0 + \alpha_1 e_{t-1}^2 + \alpha_2 e_{t-2}^2 + ... + \alpha_k e_{t-k}^2 + u_t$$

where *e* are error terms of the model in question. We check whether lagged error squares are jointly significant: the null $H_0 : \alpha_1 = \alpha_2 = ... = \alpha_k = 0$ corresponds to lack of the ARCH effect. Under the null, the test statistic is asymptotically distributed as $\chi^2(k)$.

```
Test statistic: LM = 307.262
  with p-value = P(Chi-Square(5) > 307.262) = 2.74944e-064
```

The null hypothesis of no ARCH effect is clearly rejected.

7. **The ARCH-GARCH models for logarithmic returns, with indices returns as additional explanatory variables**:

a) We estimate the ARCH model for logarithmic returns of USDPLN exchange rate[12]:
b) The GARCH model requires smaller number of parameters,
c) We compare forecasts from the GARCH model and GARCH with stock indices returns as additional variables[13]:

[11] Robert Engle: the winner of the 2003 Nobel Memorial Prize in Economic Sciences, sharing the award with Clive Granger, "for methods of analyzing economic time series with time-varying volatility (ARCH)". Has B.S. in physics from Williams College, and M.S. in physics and PhD in economics from Cornell University.

[12] R.Engle, Autoregressive Conditional Heteroskedasticity With Estimates of the Variance of UK Inflation, Econometrica 50 (1982): 987-1008.

[13] Bollerslev, Tim (1986). "Generalized Autoregressive Conditional Heteroskedasticity". Journal of Econometrics 31 (3): 307–327.
Bollerslev, Tim (1987). "A Conditional Heteroskedastic Time Series Model for Speculative Prices and Rates of Return". The Review of Economics and Statistics 69 (3): 542–547.
Bollerslev, Tim (1990). "Modeling the Coherence in Short-run Nominal Exchange Rates: A Multivariate Generalized ARCH Model". The Review of Economics and Statistics 72 (3): 498–505.
Bollerslev, Tim (1992). "ARCH Modeling in Finance: A Review of the Theory and Empirical Evidence". Journal of Econometrics 52 (1-2): 5–59.

a) The ARCH model requires greater number of lags in conditional variance equation:

```
Model: WLS (ARCH), using observations 2000/01/13-2010/11/19 (T =
2656)
Dependent variable: rlUSD
Variable used as weight: 1/sigma

             coefficient   std. error   t-ratio    p-value
  ---------------------------------------------------------
  rlUSD_1      0.0748882    0.0216649     3.457    0.0006    ***
  rlUSD_2      0.0416609    0.0216561     1.924    0.0545    *

  alpha(0)     0.314319     0.0456291     6.889    7.01e-012 ***
  alpha(1)     0.111278     0.0191041     5.825    6.41e-09  ***
  alpha(2)     0.112016     0.0188105     5.955    2.94e-09  ***
  alpha(3)     0.0223747    0.0189307     1.182    0.2373
  alpha(4)     0.204744     0.0188193    10.88     5.32e-027 ***
  alpha(5)     0.181330     0.0191122     9.488    5.07e-021 ***
```

b) The generalized ARCH model, GARCH, seems to be better solution, as it needs as a rule 1 lagged squared error and 1 lag of conditional variance:

```
Model 33: GARCH, using observations 2000/01/06-2010/11/19 (T = 2661)
Dependent variable: rlUSD
Standard errors based on Hessian

             coefficient   std. error     z       p-value
  --------------------------------------------------------
  rlUSD_1      0.0702018    0.0204575     3.432   0.0006    ***
  rlUSD_2      0.0354072    0.0204281     1.733   0.0830    *

  alpha(0)     0.0121354    0.00283214    4.285   1.83e-05  ***
  alpha(1)     0.0790725    0.0105393     7.503   6.25e-014 ***
  beta(1)      0.905264     0.0118621    76.32    0.0000    ***

Mean dependent var  -0.013443   S.D. dependent var   0.924162
Log-likelihood      -3141.544   Akaike criterion     6295.088
Schwarz criterion    6330.406   Hannan-Quinn         6307.869
```

c) Next we estimate similar GARCH model with stock indices returns as additional variables in a mean equation, and compare accuracy of forecasts for last month's data:

Model GARCH	Without stock indices	With stock indices
Mean Error	0.1779	0.1769
Mean Squared Error	1.3414	1.2460
Root Mean Squared Error	1.1582	1.1163
Mean Absolute Error	0.9747	0.9480
Bias proportion, UM	0.0236	0.0251
Regression proportion, UR	0.1431	0.2625
Disturbance proportion, UD	0.8333	0.7124

What shall we do about lack of cointegration?

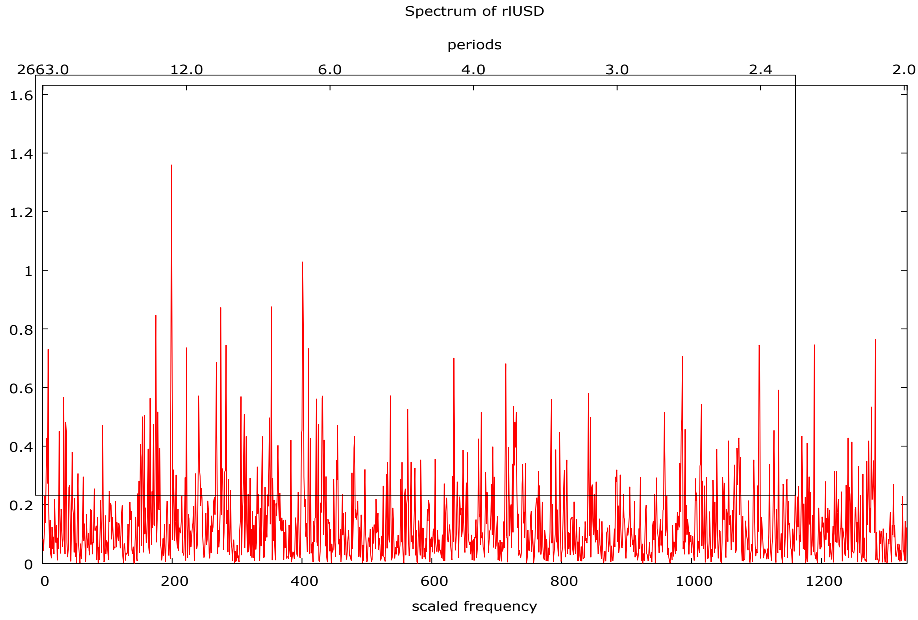
One possibility: use the Hansen method and tests[14]:
The FM-OLS method of estimation can be applied to models with structural change at unknown moment – the estimator is semiparametric, uses *spectral analysis tools to estimate long-run variance* of residuals and to use it for improving quality of estimation;

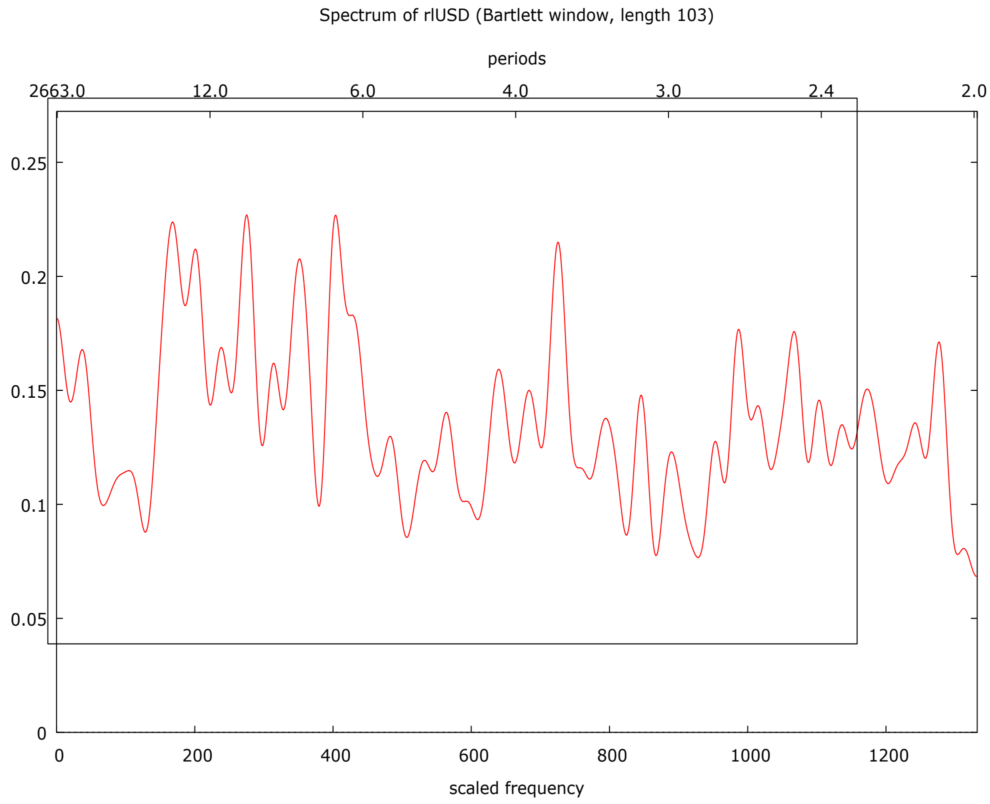
The Hansen tests, AvgF, SupF and Lc test, can be used to check whether there is *a changing-parameter cointegration relationship*. Are based on Richard Quandts idea of testing structural break at unknown point in time –we should compute values of the F-test statistic for all possible moments of break in (15%N, 85%N), N – number of observations[15].

[14] Gregory, Allan W. & Hansen, Bruce E., 1996. "Residual-based tests for cointegration in models with regime shifts," Journal of Econometrics, Elsevier, vol. 70(1), pages 99-126, January; Hansen, Bruce E, 1996. "Inference When a Nuisance Parameter Is Not Identified under the Null Hypothesis," Econometrica, Econometric Society, vol. 64(2), pages 413-30, March; Hansen, Bruce E, 1992. "Tests for Parameter Instability in Regressions with I(1) Processes," Journal of Business & Economic Statistics, American Statistical Association, vol. 10(3), pages 321-35, July.

[15] Andrews, Donald W K, 1993. "Tests for Parameter Instability and Structural Change with Unknown Change Point," Econometrica; Andrews, Donald W K, 1991. "Heteroskedasticity and Autocorrelation Consistent Covariance Matrix Estimation," Econometrica, Econometric Society, vol. 59(3), pages 817-58, May.

…
Spectral density function corresponds to long-run variance of a process, periodogram ordinates measure input to the whole variance of a particular frequency.





Thank you for kind attention