Optimal Marketing Policy in a Random Network

Hamed Amini
INRIA-ENS
45 rue d'Ulm, Paris, France
hamed.amini@ens.fr

Marc Lelarge
INRIA-ENS
45 rue d'Ulm, Paris, France
marc.lelarge@ens.fr

Abstract

Viral marketing takes advantage of preexisting social networks among customers to achieve large changes in behavior. Models of influence spread have been studied in a number of domains, including the effect of 'word of mouth' in the promotion of new products or the diffusion of technologies. A social network can be represented by a graph where the nodes are individuals and the edges indicate a form of social relationship. The flow of influence through this network can be thought of as an increasing process of active nodes: as individuals become aware of new technologies, they have the potential to pass them on to their neighbors. The goal of marketing is to trigger a large cascade of adoptions.

In this paper, we present and solve an analytical model where the individuals are connected according to a large sparse random graph. Borrowing ideas and techniques from Markov random fields, we derive analytical results for various threshold models. The parameters of the model (like the initial fraction of active individuals) are tuned by the marketer at a cost. We optimize the amount of marketing funds spent and compare marketing strategies like targeting nodes or edges. Our model allows to compute the advertising effectiveness as a function of the level of advertising spending and it shows that there is a sharp threshold phenomenon which marketing strategies should take advantage of.

1 Introduction

With consumers showing increasing resistance to traditional forms of advertising, marketers have turned to alternate strategies like viral marketing.

Viral marketing exploits existing social networks by encouraging customers to share product information with their friends. Social networks are graphs in which nodes represent individuals and edges represent relations between them. To illustrate viral marketing (see [18]), consider a company that wishes to promote its new instant messenger (IM) system. A promising way would be through popular social network such as Myspace: by convincing several persons to adopt the new IM system, the company can obtain an effective marketing campaign and diffuse the new system over the network.

If we assume that 'convincing' a person to 'spread' the new technology costs money, then a natural problem is to detect the influential members of the network who can trigger a cascade of influence in the most effective way [8], [14]. In this paper, we consider a slightly different problem: the marketer has no knowledge of the social network. Hence he will not be able to detect the most influential individuals and his only solution is to 'convince' a fraction of the total population. However, the marketer can still use the structure of the underlying network by targeting the neighbors of the adopters. There are a number of incentive programs around this idea: each time an individual chooses the new technology, he is given the opportunity to send e-mail to friends with a special offer; if the friend goes on to buy it, each of the individuals receive a small cash bonus.

The marketer is then facing an optimization problem where he can choose the fraction of population to 'convince' and then how much to spend on rebates and bonuses. In this paper, we develop a mathematical model that allows to analyze the dynamics of the cascading sequence of nodes switching to the new technology as a function of these parameters. Using idea from Markov random fields, we associate to the diffusion process a Local Mean Field (LMF) model. If the underlying social graph is sparse, we show that our LMF is an asymptotically correct approximation of the real diffusion process. Hence the LMF allows to compute the fraction of adopters at each time and also captures the local correlation among individuals. In particular, we show that our method allows to correct the standard mean field approximation.

Our work is also related to recent literature in computer science which examines games played on a network, see for example the graphical games analyzed in [13]. The graphical games literature has focused on finding efficient algorithms to compute Nash equilibria. Recently [7] described a mapping from any graphical game to a Markov random field (MRF) depending on a parameter ϵ such that for small ϵ most of the probability mass is concentrated on the set of pure Nash equilibria (if such equilibria exist). This reduction allows then to use statistical inference algorithms (like belief propagation) to compute pure Nash equilibria. The LMF introduced in this paper has a similar flavor. It can be interpreted as a message passing algorithm and its correctness is due to the locally tree like structure of the underlying graph. Finally such an approach has been recently carried out for a different game, where interactions global in [16].

The rest of the paper is organized as follows. In Section 2, we describe the model of diffusion and define the corresponding optimization problem for the marketer. In Section 3, we give a precise analysis of the diffusion process on regular random graphs. In Section 4, we define the Local Mean Field model for graphs with asymptotic given degree distribution. Finally in Section 5, we give some applications: we show that a threshold phenomenon occurs and we solve our optimization problem.

2 Model

Let $G = (V, E)$ be a graph on the vertex set $V = [1, n]$. Agents are represented by vertices of the graph. For $i, j \in V$, we write $i \sim j$ if $(i, j) \in E$ and we say that agents i and j are neighbors.

We begin by discussing one of the most basic game-theoretic diffusion models proposed by Morris [19].

- 2 possible strategies A and B.
- if 2 neighbors choose A, they each receive a payoff of q_A. If they choose B, they receive q_B and if they choose opposite strategies, then they receive a payoff of 0.
- the total payoff of a player is the sum of the payoffs with each of his neighbors.

Note that B is the 'better' technology if $q_A < q_B$, in the sense that $B - B$ payoffs exceed $A - A$ payoffs.

We consider a network G and let all nodes initially play A. If a small number of nodes are forced to adopt strategy B and we apply best-response updates to other nodes in the network, then these nodes will be repeatedly applying the following rule: switch to B if enough of your neighbors have already adopted B. There can be a cascading sequence of nodes switching to B such that a network-wide equilibrium is reached in the limit. This equilibrium may involve uniformity with all nodes adopting B or it may involve coexistence, with the nodes partitioned into a set adopting B and a set sticking to A. Morris [19] consider the case of infinite graph G and provides graph-theoretic characterizations for when these different types of equilibria arise.

The state of agent i is represented by X_i; $X_i = 0$ if player i plays strategy A and $X_i = 1$ otherwise. Hence $\sum_{j\sim i} X_j$ is the number of neighbors of i playing strategy B and $\sum_{j\sim i}(1-X_j)$ is the number of neighbors of i playing strategy A.

We now describe the economic model for the agents. As described in the introduction, the payoff for a $A-A$ edge is q_A, for a $B-B$ edge is q_B and for a $A-B$ edge is 0. Now the total payoff for an agent is given by

$$\begin{aligned} S_i^A &= q_A \sum_{j\sim i}(1-X_j) \text{ for strategy } A, \\ S_i^B &= r + (q_B+u)\sum_{j\sim i} X_j \text{ for strategy } B. \end{aligned}$$

In the first case, the payoff of the agent is just the sum of the payoffs obtained on each of his incident edges. In the second case, the payoff is the sum of these payoffs increased by an amount $u \geq 0$ plus a bonus of $r \geq 0$.

We now explain the dynamics of our model for the spread of strategy B in the network as time t evolves. We consider a fixed network G (not evolving in time) and let all agents play A for $t < 0$. At time $t = 0$, some agents are forced to strategy B. These agents will always play strategy B, hence the dynamics described below does not apply to these initially forced agents. We encode the initial population forced to strategy B by a vector χ, where $\chi_i = 1$ if agent i is forced to B and $\chi_i = 0$ otherwise. We will assume that the vector $\chi = (\chi_i)_{i\in\mathbb{N}}$ is a sequence of i.i.d. Bernoulli random variables with parameter α.

The state of the network at time t is described by the vector $(X_i(t))$. We have $X_i(0) = \chi_i$ and $X_i(t) \geq \chi_i$. Then at each time step $t \geq 1$, each agent applies the best-response update: if $S_i^B > S_i^A$ then he chooses B and if not then he chooses A.

Lemma 1. *This model is monotone in the sense that each agent changes his strategy at most once in the whole game.*

We can interpret r as a rebate done to new agents for strategy B and u as a bonus given at each pair of adopters of B. Hence we can now define the price of a marketing strategy. We consider that q_A and q_B are fixed and correspond to the level of performance of the technologies A and B. There are now three parameters to optimize:

- the price incurred in order to initially force some agents to strategy B;
- the price of the rebates r;

- the price of the bonus u.

Let $\alpha = \frac{\sum_i \chi_i}{n}$ be the proportion of forced agents. We denote by $M_1(\alpha)$ the price incurred to force the initial agents. Typically if there is a fixed cost per agent say c, we could take $M_1(\alpha) = c\alpha$. Let $\beta(t)$ be the proportion of agents choosing B at time t: $\beta(t) = \frac{\sum_i X_i(t)}{n}$. We have $\gamma(t) = \beta(t) - \alpha \geq 0$ which corresponds to the proportion of agents choosing B without being initially forced. We denote by $M_2(\gamma(t))$ the price incurred by the rebates until time t. We typically take $M_2(\gamma) = r\gamma$. Let $\delta(t)$ be the proportion of edges $B - B$ at time t. We denote by $M_3(\delta(t))$ the price incurred by the marketing of edges until time t. We typically take $M_3(\delta) = u\delta$. Hence the total price of the marketing strategy at time t is given by $M(t) = M_1(\alpha) + M_2(\gamma(t)) + M_3(\delta(t))$. The main goal of this paper is to compute the quantities $\gamma(t)$ and $\delta(t)$ in function of α, r and u. In particular, the LMF introduced in Section 4 allows to compute these quantities as t varies. This opens the possibilities of doing an optimal control of the marketing policy. We leave these issues to future research. In this paper, we will consider the case where α, r and u are fixed and optimized in order to contaminate the whole population (see Section 5).

3 Diffusion on a random regular graph

In this section we analyze our model on a large random Δ-regular graph $G_\Delta^{(n)}$ selected uniformly at random from the set of all Δ-regular graphs on n nodes (graphs in which every node has degree Δ). We have

$$1 - X_i(t+1) \quad = \quad (1 - \chi_i) \mathbb{1}\left(S_i^B(t) \leq S_i^A(t)\right). \tag{1}$$

Note that for Δ regular graphs, we have $S_i^B \leq S_i^A$ iff

$$\begin{aligned} r + (q_B + u) \sum_j X_j \leq q_A \sum_j (1 - X_j) \\ \Leftrightarrow \sum_j X_j \leq \theta(\Delta), \end{aligned}$$

with $\theta(d) := \frac{q_A d - r}{q_A + q_B + u}$.

It is well-known that locally the graph looks like a Δ-regular tree with high probability. Hence we start our analysis by looking at the regular tree.

3.1 Diffusion process on the infinite regular tree

Let $T(\Delta)$ be an infinite Δ-regular tree with nodes $\varnothing, 1, 2, \ldots$, with a fixed root $\varnothing$. For a node i, we denote by $\mathrm{gen}(i) \in \mathbb{N}$ the generation of i, i.e. the length of the minimal path from $\emptyset$ to i. Also we denote $i \to j$ if i belongs to the children of j, i.e. $\mathrm{gen}(i) = \mathrm{gen}(j) + 1$ and j is on the minimal path from $\emptyset$ to i. For an edge (i, j) with $i \to j$, we denote by $T_{i\to j}$ the sub-tree of T with root i obtained by the deletion of edge (i, j) from T.

For a given vector χ, we say that node $i \neq \emptyset$ is infected from $T_{i\to j}$ if the node i switches to B in $T_{i\to j} \bigcup \{(i, j)\}$ with the same vector χ for $T_{i\to j}$ and the strategy A for j. We denote by $Y_i(t)$ the corresponding indicator function with value 1 if i is infected from $T_{i\to j}$ at time t and 0 otherwise.

Lemma 2. *We have*

$$1 - X_\emptyset(t + 1) = (1 - \chi_\emptyset)\mathbb{1}\left(\sum_{i\sim\emptyset} Y_i(t) \leq \theta(\Delta)\right). \qquad (2)$$

Proof. We have

$$X_\emptyset(t + 1) = 1 - (1 - \chi_\emptyset)\mathbb{1}\left(\sum_{i\sim\emptyset} X_i(t) \leq \theta(\Delta)\right),$$

and let

$$Z_\emptyset(t + 1) = 1 - (1 - \chi_\emptyset)\mathbb{1}\left(\sum_{i\sim\emptyset} Y_i(t) \leq \theta(\Delta)\right),$$

now we show $\forall t \geq 0 : \ X_\emptyset(t) = Z_\emptyset(t)$ which is clear if $\chi_\emptyset = 1$. Suppose now that $\chi_\emptyset = 0$, hence $X_\emptyset(0) = Z_\emptyset(0) = 0$. By definition of Y_i we have $\forall i \sim \emptyset : \ Y_i(t) \leq X_i(t)$ and then $Z_\emptyset(t + 1) \leq X_\emptyset(t + 1)$. Hence we need to show $Z_\emptyset(t+1) \geq X_\emptyset(t+1)$. Suppose it is false and consider the first time s that the inequality is violated, i.e. $X_\emptyset(s) = 1, \ Z_\emptyset(s) = 0$. Since s is the first time that it happens we have $X_\emptyset(s - 2) = 0$ and then by definition of Y_i we will have $\forall i \sim \emptyset : \ Y_i(s - 1) = X_i(s - 1)$ which implies $X_\emptyset(s) = Z_\emptyset(s)$. $\square$

The representation (2) is crucial to our analysis because, thanks to the tree structure, the random variables $(Y_i(t), i \sim \emptyset)$ are independent of each other and identically distributed. More precisely, a simple induction shows that (1) becomes, for $i \neq \emptyset$:

$$1 - Y_i(t + 1) \;\; = \;\; (1 - \chi_i)\mathbb{1}\left(\sum_{j\to i} Y_j(t) \leq \theta(\Delta)\right). \qquad (3)$$

Note that (3) allows to compute all the $Y_i(t)$ recursively, starting with $Y_i(0) = \chi_i$. Hence a simple induction on t shows that the random variables $(Y_i(t), i \sim$

$\emptyset$) are independent of each other. It is then easy to compute their distribution from (3). We summarize this result in the next proposition.

We introduce the following notations: let $b_{k,s}(x)$

$$\begin{aligned} b_{k,s}(x) &:= \mathbb{P}\left(Bin(k,x) = \lfloor s \rfloor\right) \\ &= \binom{k}{\lfloor s \rfloor} x^{\lfloor s \rfloor}(1-x)^{k-\lfloor s \rfloor}, \end{aligned}$$

and $g_{k,s}(x)$

$$g_{k,s}(x) \quad := \quad \mathbb{P}\left(Bin(k,x) \le s\right).$$

Proposition 1. *For t fixed, the sequence $(Y_i(t), i \sim \emptyset)$ is a sequence of i.i.d. Bernoulli random variables with parameter $h(t)$ given by:*

$$h(t+1) := \mathbb{P}(Y_i(t+1) = 1) = 1 - (1-\alpha)g_{\Delta-1,\theta(\Delta)}(h(t)),$$

and $h(0) = \alpha$.

Proof. By (3) and by independence of random variables we have:

$$\begin{aligned} 1 - h(t+1) &= \mathbb{P}(Y_i(t+1) = 0) \\ &= \mathbb{P}(\chi_i = 0)\mathbb{P}(\sum_{j\sim i} Y_j(t) \le \theta(\Delta)) \\ &= (1-\alpha)g_{\Delta-1,\theta(\Delta)}(h(t)). \end{aligned}$$

□

From Lemma 2, we get directly

Corollary 1. *$X_\emptyset(t)$ is a Bernoulli random variable with parameter $\tilde{h}(t)$ given by*

$$\begin{aligned} \mathbb{P}(X_\emptyset(t+1) = 1) &= \tilde{h}(t+1) \\ &= 1 - (1-\alpha)g_{\Delta,\theta(\Delta)}(h(t)). \end{aligned}$$

It is natural to define the following fixed point equation:

$$h = 1 - (1-\alpha)g_{\Delta-1,\theta(\Delta)}(h). \tag{4}$$

We denote by h^* the smallest solution of this fixed point equation.

Lemma 3. *Suppose $0 \le \theta(\Delta) < \Delta - 2$. There exists $\alpha_{crit} < 1$ such that for all $\alpha > \alpha_{crit}$, the fixed point equation (4) has a unique solution $h^* = 1$ and for all $\alpha < \alpha_{crit}$ it has three solutions $h^* < h^{**} < h^{***} = 1$.*

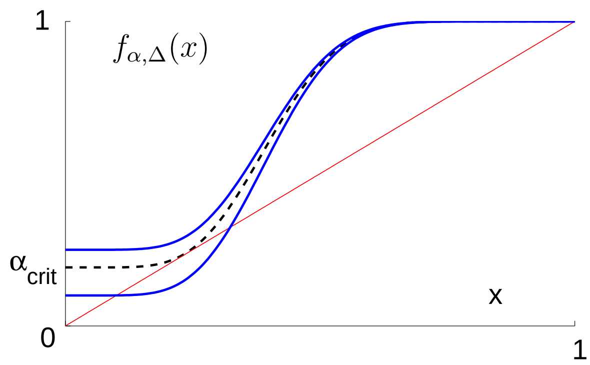

Figure 1: $f_{\alpha,\Delta}(x) = x$ has more than one solution when $\alpha > \alpha_{crit}$ ($\Delta = 20$, $\theta(\Delta) = 8$).

The proof of the Lemma is given in Appendix. The definition of α_{crit} is illustrated on Figure 1 which represents the curve of the function:

$$x \mapsto f_{\alpha,\Delta}(x) = 1 - (1-\alpha) g_{\Delta-1,\theta(\Delta)}(x), \tag{5}$$

for different values of α.

As will be explained in Section 5.1, Lemma 3 is responsible for the sharp threshold phenomenon.

Remark 1. *When $\theta(\Delta) < 1$ and $\Delta > 1$, we have*

$$g_{\Delta-1,\theta(\Delta)}(x) = \mathbb{P}(Bin(\Delta - 1, x) = 0) = (1-x)^{\Delta-1}$$

and for $\alpha > 0$ the fixed point equation (4) has a unique solution $x = 1$, i.e. $\alpha_{crit} = 0$ which is in accordance with Example 5 in [19].

We define $\tilde{h}(t+1) = 1-(1-\alpha)g_{\Delta,\theta(\Delta)}(h(t))$ and $\tilde{h} = 1-(1-\alpha)g_{\Delta,\theta(\Delta)}(h^*)$ so that

$$\mathbb{P}(X_\varnothing(t) = 1) = \tilde{h}(t) \quad \text{and} \quad \lim_{t\to\infty} \mathbb{P}(X_\varnothing(t) = 1) = \tilde{h}. \tag{6}$$

3.2 Rooting at edges instead of nodes

In this section, we still consider the infinite Δ-regular tree $T(\Delta)$ but rooted at an edge (a, b). We compute the law of the couple $(X_a(t), X_b(t))$. We follow the same argument as in previous section but for two coupled trees: T^a and

T^b which are rooted at a and b respectively. We denote with a superscript a the quantities associated to T^a and with a superscript b those associated to T^b. For $i \in T$, we have $\text{gen}^a(i) \neq \text{gen}^b(i)$, $T^a_{i\to j} \neq T^b_{i\to j}$ and so on. We denote by $Y^a_i(t)$ (resp. $Y^b_i(t)$) the indicator function with value 1 if i is infected from $T^a_{i\to j}$ (resp. $T^b_{i\to j}$) at time t and 0 otherwise. Equation (2) becomes

$$\begin{aligned} 1 - X_a(t+1) &= (1-\chi_a)\mathbb{1}\left(\sum_{i\sim a} Y^a_i(t) \leq \theta(\Delta)\right) \\ &= (1-\chi_a)\mathbb{1}\left(\sum_{i\sim a, i\neq b} Y^a_i(t) + Y^a_b(t) \leq \theta(\Delta)\right), \end{aligned} \tag{7}$$

and a similar equation holds for X_b. By the same argument as in previous section, the random variables $(Y^a_i(t), i \sim a, i \neq b)$ and $(Y^b_i(t), i \sim b, i \neq a)$ are independent Bernoulli random variables with parameter $h(t)$.

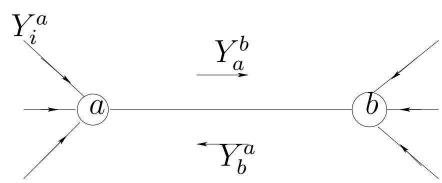

Figure 2: Process Y viewed as messages

Now (3) gives

$$\begin{aligned} 1 - Y^a_b(t+1) &= (1-\chi_b)\mathbb{1}\left(\sum_{i\to b} Y^a_i(t) \leq \theta(\Delta)\right) \\ &= (1-\chi_b)\mathbb{1}\left(\sum_{i\sim b, i\neq a} Y^b_i(t) \leq \theta(\Delta)\right) \end{aligned} \tag{8}$$

and a similar expression for Y^b_a. Hence (7) and (8) gives an explicit expression for $(X_a(t), X_b(t))$ in function of the independent random variables $(Y^a_i(t), i \sim a, i \neq b)$ and $(Y^b_i(t), i \sim b, i \neq a)$. For $a \sim b$, we denote

$$h_{B-B}(t) := \mathbb{E}[X_a(t)X_b(t)] \tag{9}$$

and its limit by

$$h_{B-B} := \lim_{t\to\infty} h_{B-B}(t). \tag{10}$$

Proposition 2. *We have*

$$h_{B-B} = F_{B-B}(h^*)$$

with

$$\begin{aligned} F_{B-B}(x) &= \alpha^2 + 2\alpha(1-\alpha)(1-g_{\Delta-1,\theta(\Delta)-1}(x)) \\ &+ (1-\alpha)^2 \left[(1-g_{\Delta-1,\theta(\Delta)}(x))^2 \right. \\ &+ \left. 2b_{\Delta-1,\theta(\Delta)}(x)(1-g_{\Delta-1,\theta(\Delta)}(x))\right]. \end{aligned} \tag{11}$$

Remark 2. *We have*

$$\begin{aligned} h_{B-B} - \tilde{h}^2 &= h_{B-B} - \left(\alpha + (1-\alpha)(1-g_{\Delta,\theta(\Delta)}(h^*))\right)^2 \\ &= 2\alpha(1-\alpha)\left(g_{\Delta,\theta(\Delta)}(h^*) - g_{\Delta-1,\theta(\Delta)-1}(h^*)\right) \\ &+ 2b_{\Delta-1,\theta(\Delta)}(h^*)(1-g_{\Delta-1,\theta(\Delta)}(h^*)) \end{aligned}$$

which is always positive because

$$g_{\Delta,\theta(\Delta)}(h^*) \geq g_{\Delta-1,\theta(\Delta)-1}(h^*).$$

3.3 Random Regular Graphs

We now come back to the process $(X_i^{(n)}(t), i \in [1,n])$ on $G_\Delta^{(n)}$ satisfying (1). Our first result shows that the process defined on the tree in Section 3.1 is a good approximation of the real process since we have the following asymptotics:

Proposition 3. *We have for any fixed $t \geq 0$, and $i \sim j$,*

$$\lim_{n\to\infty} \mathbb{E}\left[X_i^{(n)}(t)\right] = \tilde{h}(t), \tag{12}$$

$$\lim_{n\to\infty} \mathbb{E}\left[X_i^{(n)}(t)X_j^{(n)}(t)\right] = h_{B-B}(t), \tag{13}$$

where $\tilde{h}(t)$ is defined in (6) and $h_{B-B}(t)$ is defined in (9).

Proof. We only prove (31). The proof of (13) follows the same argument. Clearly by symmetry, we need to prove the statement for $i = 0$ only. Given $d \geq 1$, let $N(i,d,G_\Delta^{(n)})$ be the set of vertices of $G_\Delta^{(n)}$ that are at a distance at most d from $i \in G_\Delta^{(n)}$. A depth-d Δ-regular tree $T(\Delta,d)$ is the restriction of $T(\Delta)$ to nodes i with $\mathrm{gen}(i) \leq d$. A simple induction on t shows that $X_i^{(n)}(t)$ is determined by the $\{\chi_j, j \in N(i,t,G_\Delta^{(n)})\}$. The statement then follows from the following convergence [12]: for any fixed $d \geq 1$, we have as n tends to infinity $N(0,d,G_\Delta^n) \xrightarrow{d} T(\Delta,d)$. Hence we have $X_0^{(n)}(t) \xrightarrow{d} X_\emptyset(t)$ as n tends to infinity and the proposition follows. □

As a consequence of Proposition 7, we are able to compute the asymptotics (in n) statistics of the process $(X_i^{(n)}(t), i \in [1,n])$. Let $\beta^{(n)}(t)$ be the proportion of agents choosing B at time t: $\beta^{(n)}(t) = \frac{\sum_i X_i^{(n)}(t)}{n}$. We have as $n \to \infty$,

$$\mathbb{E}\left[\beta^{(n)}(t)\right] = \mathbb{E}\left[X_i^{(n)}(t)\right] \to \tilde{h}(t). \tag{14}$$

The final proportion of agents choosing B is $\beta^{(n)} = \lim_{t\to\infty} \beta^{(n)}(t)$.

Proposition 4. *We have*

$$\lim_{n\to\infty} \mathbb{E}\left[\beta^{(n)}\right] = \tilde{h}, \tag{15}$$

in particular, for $\alpha \geq \alpha_{crit}$, we have

$$\lim_{n\to\infty} \mathbb{E}\left[\beta^{(n)}\right] = 1. \tag{16}$$

Remark 3. *Proposition 4 does not follow from Proposition 7 and the interchange of limits in t and n needs a proper mathematical proof. This has been done in [6] and our statement follows from their Theorem 1. For Δ-regular graphs, bootstrap percolation is equivalent to our model. It is noticed in [6] that the critical value on the Δ-regular random graph turns out to be the same as that on the Δ-tree (computed in [5]), although the proof goes along a quite different route.*

Let $\delta^{(n)}(t)$ be the proportion of $B-B$ edges at time t:

$$\delta^{(n)}(t) = \frac{2\sum_{i\sim j, i<j} X_i^{(n)}(t) X_j^{(n)}(t)}{n\Delta}.$$

We have as $n \to \infty$,

$$\mathbb{E}\left[\delta^{(n)}(t)\right] \to h_{B-B}(t). \tag{17}$$

The final proportion of edges $B-B$ is $\delta^{(n)} = \lim_{t\to\infty} \delta^{(n)}(t)$. Similarly as Proposition 4, we have

Proposition 5. *We have*

$$\lim_{n\to\infty} \mathbb{E}\left[\delta^{(n)}\right] = h_{B-B}, \tag{18}$$

in particular, for $\alpha \geq \alpha_{crit}$, we have

$$\lim_{n\to\infty} \mathbb{E}\left[\delta^{(n)}\right] = 1. \tag{19}$$

4 A Local Mean Field model for Diffusion on Random Networks

This section relates our model to recent results of the economic literature. We will show how the analysis made in previous section carries over to sparse random graphs with asymptotic given degree distribution $P(d)$ (see [9]). We will see that our analysis allows to correct the mean-field approximation used recently to model diffusion of behavior by taking into account a local correlation. We also introduce a Markov Random Fields associated to our diffusion model that we call the Local Mean Field (LMF) model. It extends the standard mean-field approximation by allowing to model the correlation structure on local neighborhoods.

4.1 Mean Field Approximation for Diffusion

The study of games on networks has attracted a lot of attention during the recent year. We refer to [10] for the description of a general framework and references to the economic literature. An attractive way to model the social network consists in using a mean field approach to study the random network. The main advantage of this approach is its simplicity. A random network is characterized by its connectivity distribution: the fraction of agents in the population with d neighbors is described by the degree distribution $P(d)$. Using such a framework [17] computes the contagion threshold on random networks for the contagion process introduced by Morris [19]. A similar model is analyzed in [11], where the dynamic of the diffusion process is also considered. The heuristic argument goes as follows [11]: let $\tilde{P}(d) = \frac{dP(d)}{\sum dP(d)}$ be the probability that an edge points to a node with degree d. Assume that agents have a choice between taking an action 0 or an action 1 (action 0 being the default behavior). Let $H(d,x)$ be the probability that a random best responding neighbor of degree d chooses the action 1 when anticipating that each neighbor will choose 1 with an independent probability x. At time $t=0$, a fraction x^0 of the population is exogenously and randomly assigned the action 1. At each time, $t>0$, each agent best responds to the distribution of agents choosing the action 1 in period $t-1$, presuming that their neighbors will be random draw from the population. Then if x_d^t denotes the fraction of those agents with degree d who have adopted behavior 1 at time t, then simple calculations gives [11]

$$x^{t+1} := \sum_d \tilde{P}(d) x_d^{t+1} = \sum_d \tilde{P}(d) H(d, x^t). \tag{20}$$

The static Bayesian equilibria are given by the fixed point equation

$$x = \sum_d \tilde{P}(d) H(d, x) =: \phi(x). \tag{21}$$

These mean field equations offer an approximation of diffusion by neglecting the spatial correlation among the agents. To put it differently, these equations correspond to a model where the random network is generated every period, although the connectivity of each agent remains constant. Our LMF model allows to take into account this spatial correlation and we now show how our approach correct the mean field equations.

4.2 Comparison in the case of Δ-regular graphs

We now compute the quantities appearing in (20) and (21) and compare these equations with our results obtained in Section 3.

Clearly we have $P(d) = \tilde{P}(d) = \mathbb{1}(d = \Delta)$. Following [11], we have

$$H(d, x) \quad = \quad 1 - (1-\alpha)\mathbb{E}\left[\sum_{i=1}^{d} \mathrm{Ber}_i(x) \le \theta(d)\right],$$

where $\mathrm{Ber}_i(x)$ are i.i.d. Bernoulli random variables with parameter x. By (21), we have

$$\phi(x) = 1 - (1-\alpha) g_{\Delta, \theta(\Delta)}(x),$$

with the notation of Section 3. Hence we see that the asymptotic fraction of agents who adopt behavior 1 at equilibrium is given by x^* solution of the fixed point equation $x^* = \phi(x^*)$. This result has to be compared to Proposition 4 which states that the asymptotic fraction of agents adopting 1 is $\tilde{h}$ defined as follows:

$$\tilde{h} = 1 - (1-\alpha) g_{\Delta, \theta(\Delta)}(h^*),$$

and h^* is solution of the fixed point equation:

$$h^* = 1 - (1-\alpha) g_{\Delta-1, \theta(\Delta)}(h^*).$$

By monotonicity of the function g, we have $\tilde{h} \le x^*$. Hence the mean field approximation overestimates the size of the cascade of adoption of the new technology.

Our LMF model takes into account the spatial correlation of the agents and if the random (finite) graph is sparse, the correlation modeled in the LMF corresponds exactly to the asymptotic correlation among agents as shown by limit (13) in Proposition 7. The LMF allows us to compute exact asymptotic for sparse random graphs. We explicitly define the LMF model associated to random graphs with asymptotic given degree distribution in the next section.

4.3 Local Mean Field Model

Our LMF model is also characterized by its connectivity distribution $P(d)$ but we now construct a tree-indexed process. The underlying tree T is a Galton-Watson branching process with a root which has offspring distribution P and all other nodes have offspring distribution P^* given by $P^*(d-1) = \tilde{P}(d)$ for all $d \geq 1$.

Remark 4. *In the Δ-regular case, we have $P(d) = \mathbb{1}(d = \Delta)$ and $P^*(d) = \mathbb{1}(d = \Delta - 1)$. In this case the tree T is the Δ-regular tree $T(\Delta)$.*

Label the root as $\emptyset$ and $\mathbf{i} = i_1 i_2 \dots i_n$ denotes the nth-generation individual, the i_n-th child of its parent $i_1 i_2 \dots i_{n-1}$. Let $(\chi_{\mathbf{i}}, \mathbf{i} \in T)$ be iid Bernoulli random variable with parameter α. We have now a family of coupled trees $(T^{\mathbf{i}}, \mathbf{i} \in T)$ where $T^{\mathbf{i}}$ denotes the tree T rooted at $\mathbf{i}$. One can show that for all $\mathbf{i}, \mathbf{j}$, we have $T^{\mathbf{i}} \stackrel{d}{=} T^{\mathbf{j}}$ (this is clear in the case $T = T(\Delta)$). This property on the law of T is called unimodularity [1]. We denote by $\text{gen}^{\mathbf{i}}(\mathbf{j})$ the generation of $\mathbf{j}$ in $T^{\mathbf{i}}$ and by $\mathbf{l} \to^{\mathbf{i}} \mathbf{j}$ if $\mathbf{l}$ is a child of $\mathbf{j}$ in $T^{\mathbf{i}}$, i.e. $\text{gen}^{\mathbf{i}}(\mathbf{l}) = \text{gen}^{\mathbf{i}}(\mathbf{j}) + 1$ and $\mathbf{j}$ is on the minimal path from $\mathbf{i}$ to $\mathbf{l}$. We are now ready to define our tree-indexed process $(Y^{\mathbf{i}}_{\mathbf{j}}(t), t \in \mathbb{R})$:

$$1 - Y^{\mathbf{i}}_{\mathbf{j}}(t+1) = (1 - \chi_{\mathbf{j}}) \mathbb{1} \left(\sum_{\mathbf{l} \to^{\mathbf{i}} \mathbf{j}} Y^{\mathbf{i}}_{\mathbf{l}}(t) \leq \theta(\Delta_{\mathbf{j}}) \right), \tag{22}$$

where $\Delta_{\mathbf{j}}$ is the degree of $\mathbf{j}$ in T. Then we define the tree-indexed process $(X_{\mathbf{j}}(t), t \in \mathbb{R})$ by

$$1 - X_{\mathbf{i}}(t+1) = (1 - \chi_{\mathbf{i}}) \mathbb{1} \left(\sum_{\mathbf{l} \to^{\mathbf{i}} \mathbf{i}} Y^{\mathbf{i}}_{\mathbf{l}}(t) \leq \theta(\Delta_{\mathbf{i}}) \right). \tag{23}$$

This process is an approximation of the process defined on a large sparse graph with asymptotic degree distribution given by P. We did provide a proof in the case of random regular graphs (see Proposition 7), the general case is similar and based on the notion of local weak convergence [2].

We now show how this process allows to derive the LMF equations for the evolution of the fraction of the population adopting behavior 1. Thanks to the unimodularity property, one can show that the random variables $(X_{\mathbf{j}}(t), \mathbf{j} \in T)$ are identically distributed (but not independent). Hence we need only to consider the original tree $T = T^{\emptyset}$ and we omit the superscript $^{\emptyset}$. We have

$$\mathbb{E}\left[X_{\emptyset}(t+1)\right] = 1 - (1-\alpha)\mathbb{E}\left[\sum_{j \to \emptyset} Y_j(t) \leq \theta(\Delta_{\emptyset}) \right].$$

A simple induction argument on t shows that for any fixed $\mathbf{i}$, the random variables $(Y_{\mathbf{l}}(t), \mathbf{l} \to \mathbf{i})$ are iid Bernoulli random variables with parameter denoted by $h(t)$. Then we have

$$\begin{aligned} h(t+1) &= \mathbb{E}\left[Y_{\mathbf{i}}(t+1)\right] \\ &= 1-(1-\alpha)\mathbb{E}\left[\sum_{\mathbf{j}\to\mathbf{i}} Y_{\mathbf{j}}(t) \le \theta(\Delta_{\mathbf{i}})\right] \\ &= 1-(1-\alpha)\sum_d P^*(d) g_{d,\theta(d+1)}(h(t)), \end{aligned} \tag{24}$$

where the function $g_{k,x}$ was defined in Section 3. Then we have for $\beta(t)$ the asymptotic fraction of agents choosing 1,

$$\begin{aligned} \beta(t+1) &:= \mathbb{E}\left[X_{\emptyset}(t+1)\right] \\ &= 1-(1-\alpha)\sum_d \tilde{P}(d) g_{d,\theta(d)}(h(t)), \end{aligned} \tag{25}$$

where $h(t)$ is given by (24). Then the associated fixed point equation is:

$$h^* = 1-(1-\alpha)\sum_d P^*(d) g_{d,\theta(d+1)}(h^*), \tag{26}$$

and we have $\lim_{t\to\infty}\beta(t) = \tilde{h}$ given by

$$\tilde{h} = 1-(1-\alpha)\sum_d P(d) g_{d,\theta(d)}(h^*). \tag{27}$$

4.4 Stationary Solution of the LMF

We now give a probabilistic interpretation in term of our LMF for the quantities h^* and $\tilde{h}$ introduced in the fixed point equation (26) and (27). In view of (22), it is natural to introduce the following Recursive Distributional Equation (RDE):

$$1 - Y_j \overset{d}{=} (1-\chi)\mathbb{1}\left(\sum_{l=1}^{\Delta-1} Y_l \le \theta(\Delta)\right), \tag{28}$$

where Δ has distribution P, χ is a Bernoulli random variable with parameter α, Y and Y_l are i.i.d. copies and all random variables are independent of each others. RDE for the process Y plays a similar role as the equation $\mu = K\mu$ for the stationary distribution of a Markov chain with kernel K, see [3].

If we denote (24) as $h(t+1) = \Psi(h(t))$, then Ψ is non-decreasing and hence the sequence $\{h(t)\}_t$ is non-decreasing and converges to $h^* \leq 1$ which is a solution of the fixed point equation (26) $h = \Psi(h)$. Then it is easy to see that the Bernoulli distribution with parameter h^* is a solution of the RDE (28).

For any solution h^* of the fixed point equation (26) $h = \Psi(h)$, it is possible to construct an invariant version of the tree-indexed process on the tree T where for each $k \geq 0$ and $t \geq 0$, the sequence $(Y_{\mathbf{i}}(t), \text{gen}(\mathbf{i}) = k)$ is a sequence of i.i.d. Bernoulli random variables with parameter h^*, see [3]. Then if we denote (25) as $x^{t+1} = \Upsilon(h(t))$, we see that the associated X-process is such that for each $k \geq 0$ and $t \geq 0$, the sequence $(X_{\mathbf{i}}(t), \text{gen}(\mathbf{i}) = k)$ is a sequence of i.i.d. Bernoulli random variables with parameter $\tilde{h} = \Upsilon(h^*)$. In other words, we can associate to any solution of (26) an invariant tree-indexed process. These invariant processes are natural candidates to describe the equilibria of the game on the random network.

5 Applications

5.1 Sharp Threshold Phenomena

We first consider the case of random regular graphs. In this case Proposition 4 shows that the final proportion of agents playing B is asymptotically given by

$$\tilde{h} = 1 - (1-\alpha) g_{\Delta,\theta(\Delta)}(h^*),$$

where h^* is the first solution of the fixed point equation

$$h = 1 - (1-\alpha) g_{\Delta-1,\theta(\Delta)}(h).$$

Figure 3 represents $\tilde{h}$ and h_{B-B} as functions of α.

We see that the function $\alpha \mapsto \tilde{h}(\alpha)$ grows almost linearly for $\alpha \leq \alpha_{crit}$ and then jumps to 1 for $\alpha > \alpha_{crit}$ (see Lemma 3). This is a sharp threshold phenomenon.

This phenomenon is not restricted to random regular graphs. We take the example of Erdős-Rényi graphs. Let $G^{(n)} = G(n, \lambda/n)$ be a random graph on n nodes $\{0, 1, ..., n-1\}$, where each potential edge (i,j), $0 \leq i < j \leq n-1$ is present in the graph with probability λ/n, independently for all $n(n-1)/2$ edges. Here $\lambda > 0$ is a fixed constant independent of n. Then for any fixed d, the neighborhood of radius d about node 0, $N(0, d, G^{(n)})$ converges in

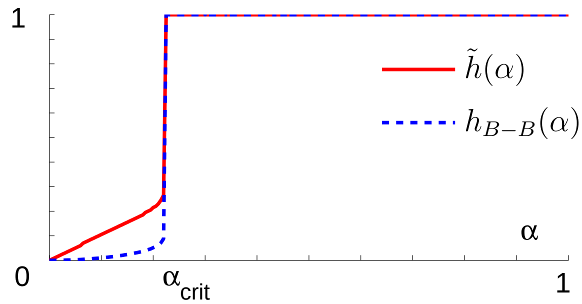

Figure 3: $\tilde{h}(\alpha)$ and $h_{B-B}(\alpha)$ ($\theta(\Delta)=8$, $\Delta=20$) .

distribution as n tends to infinity to a depth-d Poisson tree , i.e. a Galton-Watson tree with offspring distribution Poisson with parameter λ [20]. Then by (25) and (26), the fixed point equation associated to our model is:

$$h^* \;=\; 1-(1-\alpha)\sum_{k=0}^{\infty}\left(e^{-\lambda}\frac{\lambda^k}{k!}\right) g_{k,\theta(k+1)}(h^*), \tag{29}$$

and the final proportion of agents playing B is asymptotically given by

$$\tilde{h} \;=\; 1-(1-\alpha)\sum_{k=0}^{\infty}\left(e^{-\lambda}\frac{\lambda^k}{k!}\right) g_{k,\theta(k)}(h^*).$$

Note that in this case, with the notation of Section 4, we have $P=P^*$ is just the Poisson distribution with parameter λ.

In this case, the value of the right hand side of (29) is always smaller than one for $\alpha<1$ (but it can be very close to one). This means that there is always a fraction of the population who continues to play A. The intuition for this phenomenon is clear: a node having as neighbors many leaves (i.e. nodes not connected to the rest of the network) which are not forced to B will continue to play A because the incentive to switch to B will not be enough.

However we still have a threshold phenomenon as we can see in Figure 4. In the Poisson case, the value of $\alpha_{crit}(\lambda)$ can be characterized as the minimum value of α such that for $\alpha>\alpha_{crit}(\lambda)$ the fixed point equation (29) has a unique solution in $[0,1]$. We found that there is not monotonicity between the regular and the Poisson case.

The curve of Figure 4 should be compared to the response curve relating the level of advertising spending (here α) and advertising effectiveness (here

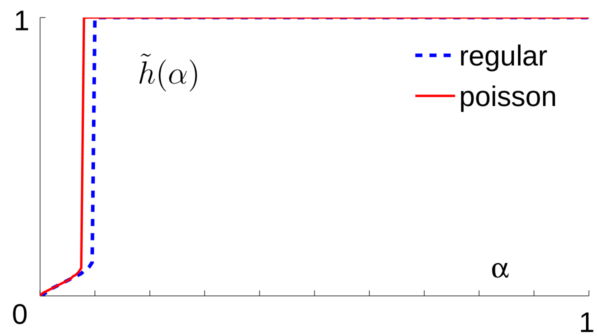

Figure 4: Threshold phenomenon in Poisson and regular tree ($\theta(\Delta) = 4$, $\lambda = \Delta = 15$) .

the final fraction of adopters). The S-shaped curve captures both the phenomena of increasing and decreasing marginal returns to the various levels of advertising spending [4]. The point α_{crit}, point of inflection, splits the curve into two parts, the bottom part being convex (representing increasing marginal returns) and the upper part being concave (representing decreasing marginal returns). It plays a crucial role in marketing strategy and we look at one example in the next section.

5.2 Optimal Marketing Policy

We now come back to our optimization problem stated in section 2.

$$M(t) \quad = \quad M_1(\alpha) + M_2(\gamma(t)) + M_3(\delta(t)).$$

Our goal is to minimize this function such that B becomes epidemic, i.e. $\tilde{h} = 1$. We consider the case of random regular graphs, where Δ and q are fixed. Then with the notation introduced in Section 3, we have $M_1 = c\alpha$, $M_2(t) = r(\tilde{h}(t) - \alpha)$ and $M_3(t) = u\frac{\Delta}{2}(h_{B-B}(t) - \alpha^2)$. Hence

$$M(t) \quad = \quad c\alpha + r(\tilde{h}(t) - \alpha) + u\frac{\Delta}{2}(h_{B-B}(t) - \alpha^2).$$

We consider the limit case where t goes to infinity. Let us denote by $M(r, u, \alpha)$:

$$M(r, u, \alpha) \quad := \quad \lim_{t \to \infty} M(t)$$

The condition $\tilde{h} = 1$ is equivalent to $\alpha \geq \alpha_{crit}$. So we face the following problem

$$\begin{cases} \inf_{r,u,\alpha} c\alpha + r(1-\alpha) + u\frac{\Delta}{2}(1-\alpha^2) \\ \text{such that} : \alpha \geq \alpha_{crit}(r,u) \end{cases}$$

Let us denote the value of this problem by M_{Opt}.

If we fix r and u, because $M(r,u,\alpha)$ is a concave function of α, under the condition $\alpha \geq \alpha_{crit}(r,u)$, it reaches its minimum at $\alpha = \alpha_{crit}(r,u)$ or at $\alpha = 1$. For $\alpha = 1$ we have

$$\inf_{r,u} M(r,u,1) \quad = \quad c$$

Hence

$$M_{Opt} = Min\{\inf_{r,u} M(r,u,\alpha_{crit}(r,u)), c\}$$

where

$$\begin{aligned} M(r,u,\alpha_{crit}(r,u)) \quad &= \quad c\alpha_{crit}(r,u) + r(1-\alpha_{crit}(r,u)) \\ &+ \quad u\frac{\Delta}{2}(1-\alpha_{crit}(r,u)^2). \end{aligned}$$

Figure 5 shows α_{crit} as a function of r and u.

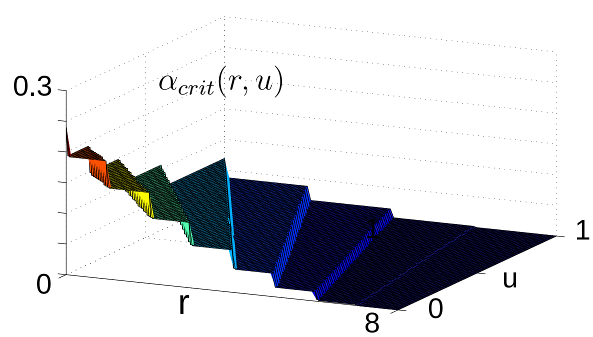

Figure 5: $\alpha_{crit}(r,u)$ ($\Delta = 20$, $q_A = 0.4$, $q_B = 0.6$) .

This problem can be solved for different values of c. Figure 6 shows $M(r,u,\alpha_{crit}(r,u))$ as a function of r and u.

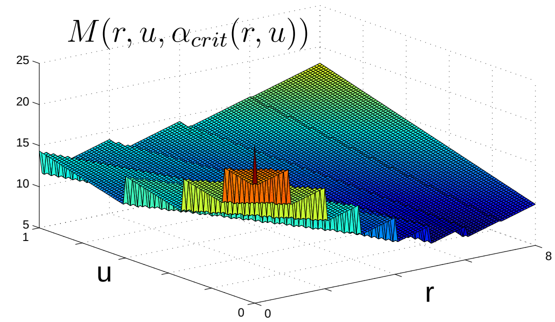

Figure 6: $M(r, u, \alpha_{crit}(r, u))$ ($\Delta = 20$, $q_A = 0.4$, $q_B = 0.6$, $c = 100$) .

5.3 Extension to General Threshold Model

The method developed in the last sections is not restricted to this particular model. We consider the general threshold model (see [15]). We have a non-negative random weight W_{ij} on each edge, indicating the influence that i exerts on j. We consider the symmetric case where $W_{ij} = W_{ji}$ and we assume W_{ij} are i.i.d with distribution function W.

Each node i has an arbitrary function f_i defined on subsets of its neighbor set N_i: for any set of neighbors $X \subseteq N_i$, there is a value $f_i(X)$ between 0 and 1 which is monotone in the sense that if $X \subseteq Y$, then $f_i(X) \leq f_i(Y)$. This node choose a threshold θ_i at random from $[0, 1]$ and at time step $t+1$ it becomes active, it plays B, if its set of currently active neighbors $N_i^B(t)$ satisfies $f_i(N_i^B(t)) > \theta_i$.

$$N_i^B(t) := \{j \sim i \quad | \quad X_j(t) = 1\}$$

We take the same definition for X_i, χ_i and Y_i, so we have the following dynamic for our system:

$$1 - X_i(t+1) = (1 - \chi_i)\mathbb{1}\left(f_i(N_i^B(t)) \leq \theta_i\right). \tag{30}$$

We will assume here that $f_i(X)$ is a function of d_i the degree of i and W_{ij} for $j \in X$, so it can be written as $f_{d_i}(W_{ij}, j \in X)$. Let $G_{d,k}$ denotes

$$G_{d,k} := \mathbb{E}^\theta \left[\mathbb{P}(f_d(W_1, ..., W_k) \leq \theta)\right]$$

where $W_1, ..., W_k$ are i.i.d with distribution function W.

Let $\tilde{N}_i^B(t)$ denotes

$$\tilde{N}_i^B(t) := \{j \to i \quad | \quad Y_j(t) = 1\},$$

so we have

$$1 - Y_i(t+1) \quad = \quad (1-\chi_i)\mathbb{1}\left(f_i(\tilde{N}_i^B(t)) \leq \theta_i\right).$$

Proposition 6. *For t fixed, the sequence $(Y_i(t), i \sim \emptyset)$ is a sequence of i.i.d. Bernoulli random variables with parameter $h(t)$ given by:*

$$\begin{aligned}
\mathbb{P}(Y_i(t+1) = 1) &= h(t+1) \\
&= 1 - (1-\alpha)\sum_d P^*(d) \sum_{k \leq d} b_{d,k}(h(t)) G_{d+1,k}
\end{aligned}$$

and $h(0) = \alpha$.

Corollary 2. *$X_\emptyset(t)$ is a Bernoulli random variable with parameter $\tilde{h}(t)$ given by*

$$\begin{aligned}
\mathbb{P}(X_\emptyset(t+1) = 1) &= \tilde{h}(t+1) \\
&= 1 - (1-\alpha)\sum_d \tilde{P}(d) \sum_{k \leq d} b_{d,k}(h(t)) G_{d,k}.
\end{aligned}$$

We denote by h^* the smallest solution of the fixed point equation

$$h \quad = \quad 1 - (1-\alpha)\sum_d P^*(d) \sum_{k \leq d} b_{d,k}(h) G_{d+1,k}.$$

So we have

$$\begin{aligned}
\tilde{h} &= \lim_{t \to \infty} \tilde{h}(t) \\
&= 1 - (1-\alpha)\sum_d \tilde{P}(d) \sum_{k \leq d} b_{d,k}(h^*) G_{d,k}.
\end{aligned}$$

We now come back to the process $(X_i^{(n)}(t), i \in [1, n])$ on $G^{(n)}$:

Proposition 7. *We have for any fixed $t \geq 0$,*

$$\lim_{n \to \infty} \mathbb{E}\left[X_i^{(n)}(t)\right] \quad = \quad \tilde{h}(t). \tag{31}$$

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6 Appendix

6.1 Proof of Lemma 1

We prove that the only possible change in the strategy of an agent is from A to B. Suppose it is not correct and consider the first time t an agent changes from B to A. Let i denote this agent and Δ_i the number of neighbors of this agent. Let S_0 be the initial set of nodes forced to play B. By definition, $i \notin S_0$. Hence there is a time $s < t$ such that i changed his strategy at this time from A to B. By definition of t, the model is monotone before t and we have $\forall i \in V$:

$$X_i(s-1) \leq X_i(t-1)$$

which implies

$$\begin{aligned}
X_i(t) &= \mathbb{1}\left(r + (1-q+u)\sum_{j\sim i} X_j(t-1) > q\sum_{j\sim i}(1 - X_j(t-1))\right) \\
&= \mathbb{1}\left(\sum_{j\sim i} X_j(t-1) \leq \frac{q\Delta_i - r}{1+u}\right) \\
&\geq \mathbb{1}\left(\sum_{j\sim i} X_j(s-1) \leq \frac{q\Delta_i - r}{1+u}\right) = X_i(s)
\end{aligned}$$

which contradicts $X_i(t) = 0$ and $X_i(s) = 1$.

6.2 Proof of Lemma 3

$h = 1$ is always a solution of this equation. By definition

$$f_{\alpha,\Delta}(x) := 1 - (1-\alpha)g_{\Delta-1,\theta(\Delta)}(x).$$

With some simple calculations we find the following properties for $f_{\alpha,\Delta}(x)$:

1. $f_{\alpha,\Delta}(x)$ is an increasing function of x,
2. $f_{\alpha,\Delta}(x)$ is convex for $\left(x < \frac{\lfloor\theta(\Delta)\rfloor}{\Delta-2}\right)$ and is a concave function for $\left(x > \frac{\lfloor\theta(\Delta)\rfloor}{\Delta-2}\right)$.

Hence by these properties the three cases shown in figure(1) are possible. Let us show now that $\alpha_{crit} < 1$. Suppose $h < 1$; by (4) we have:

$$\alpha = 1 - \frac{1-h}{g_{\Delta-1,\theta(\Delta)}(h)}.$$

Let

$$\alpha^* = 1 - \inf_{x\in(0,1)} \frac{1-x}{g_{\Delta-1,\theta(\Delta)}(x)},$$

clearly for $\alpha > \alpha^*$ the fixed point equation (4) has not solution in $(0,1)$. Now it rests to show $\alpha^* < 1$. Using the fact that

$$\mathbb{P}(Bin(n,p) \leq i) = \mathbb{P}(Beta(i+1, n-i) > p)$$

we have :

$$g_{n,s}(x) \quad = \quad \frac{n!}{\lfloor s \rfloor!(n-1-\lfloor s \rfloor)!} \left(\int_x^1 u^{\lfloor s \rfloor}(1-u)^{n-\lfloor s \rfloor -1} du \right).$$

Then we use the following inequality:

$$u^a(1-u)^b \quad \leq \quad \left(\frac{a}{a+b}\right)^a \left(\frac{b}{a+b}\right)^b,$$

and we find for $\theta(\Delta) < \Delta - 2$:

$$\begin{aligned} \alpha^* \quad &\leq \quad 1 - \left(\frac{\lfloor\theta(\Delta)\rfloor!(\Delta - \lfloor\theta(\Delta)\rfloor - 2)!}{(\Delta-1)!}\right) \left(\frac{\lfloor\theta(\Delta)\rfloor}{\Delta-2}\right)^{-\lfloor\theta(\Delta)\rfloor} \left(1 - \frac{\lfloor\theta(\Delta)\rfloor}{\Delta-2}\right)^{-(\Delta-2-\lfloor\theta(\Delta)\rfloor)} \\ &< \quad 1. \end{aligned}$$

6.3 Proof of Proposition 2

By (7) and (8):

$$1 - X_a(t+1) = (1-\chi_a)\mathbb{1}\left(\sum_{i\sim a, i\neq b} Y_i^a(t) + 1 - (1-\chi_b)\mathbb{1}\left(\sum_{i\sim b, i\neq a} Y_i^b(t-1) \leq \theta(\Delta)\right) \leq \theta(\Delta)\right).$$

So when t tends to infinity we will have :

$$1 - X_a \quad = \quad (1-\chi_a)\mathbb{1}\left(\sum_{i\sim a, i\neq b} Y_i^a + 1 - (1-\chi_b)\mathbb{1}\left(\sum_{i\sim b, i\neq a} Y_i^b \leq \theta(\Delta)\right) \leq \theta(\Delta)\right)$$

and the same thing for X_b. By monotone convergence theorem we have

$$\lim_{t\to\infty} \mathbb{E}\left[X_a(t)X_b(t)\right] = \mathbb{E}\left[X_a X_b\right] = \mathbb{P}(X_a X_b = 1).$$

Then we use

$$\begin{aligned}
X_aX_b &= \chi_a\chi_b + (1-\chi_a)\chi_b \mathbb{1}\left(\sum_{i\sim a, i\neq b} Y_i^a > \theta(\Delta) - 1\right) \\
&+ \chi_a(1-\chi_b)\mathbb{1}\left(\sum_{i\sim a, i\neq a} Y_i^b > \theta(\Delta) - 1\right) \\
&+ (1-\chi_a)(1-\chi_b)\left[\mathbb{1}\left(\sum_{i\sim a, i\neq b} Y_i^a > \theta(\Delta)\right)\right. \\
&\quad \mathbb{1}\left(\sum_{i\sim b, i\neq a} Y_i^b > \theta(\Delta)\}\right) + \mathbb{1}\left(\sum_{i\sim a, i\neq b} Y_i^a > \theta(\Delta)\right) \\
&\quad \mathbb{1}\left(\sum_{i\sim b, i\neq a} Y_i^b \leq \theta(\Delta)\right)\mathbb{1}\left(\sum_{i\sim b, i\neq a} Y_i^b > \theta(\Delta) - 1\right) \\
&+ \mathbb{1}\left(\sum_{i\sim b, i\neq a} Y_i^b > \theta(\Delta)\right)\mathbb{1}\left(\sum_{i\sim a, i\neq b} Y_i^a \leq \theta(\Delta)\right) \\
&\quad \left.\mathbb{1}\left(\sum_{i\sim a, i\neq b} Y_i^a > \theta(\Delta) - 1\right)\right]
\end{aligned}$$

which implies (11).

6.4 Proof of Proposition 5

Let $I_j^{(n)}$ denotes the total count of inactive vertices with exactly j active neighbors in $G_\Delta^{(n)}$ at the end of the process; when t tends to infinity. This has been shown in [6] that for $j < \theta$ we have:

$$i_j := \lim_{n\to\infty} \frac{I_j^{(n)}}{n} = (1-\alpha)b_{\Delta,j}(h^*).$$

So we have

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}\left[\delta^{(n)}\right] &= 1 - \frac{2}{\Delta}\sum_{j<\theta(\Delta)}\left(ji_j + \frac{\Delta - j}{2} i_j\right) \\
&= 1 - (1-\alpha)\left(g_{\Delta,\theta(\Delta)}(h^*) - h^* g_{\Delta-1,\theta(\Delta)-1}(h^*)\right)
\end{aligned}$$

Now using (4) and the following equality

$$g_{\Delta,\theta}(x) = x g_{\Delta-1,\theta-1}(x) + (1-x) g_{\Delta-1,\theta}(x)$$

implies (18).