Phase transition of product locations due to social interactions of consumers ¹

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Abstract

In the present paper we propose a stochastic process to explain the evolution of socially interacting consumers deciding between two product variants over time. With this model we support and extend Becker's (1991) explanation about the effect of social interactions on prices charged by two similar restaurants competing for consumers. We show that the demand polarization proposed by Becker is just assured by the high level of similarity of product variants (the restaurants). We also show that such demand polarization does not occur when product variants are dissimilar and when consumers' preferences are sufficiently heterogeneous.

The proposed stochastic process also allows us to analyze combinations of fixed and varying consumers' heterogeneities over time. Our theoretical results suggest how to combine these types of heterogeneities in order to drive the state of consumers' decisions into a desired state of decisions.

A game theoretical explanation for the persistence of market share asymmetry is presented for duopolies with interacting consumers. This is deduced from the equilibrium analysis of a price-game, in which there are two producers - game players - and many socially interacting consumers. We also present a generalization of the price-game, the location-price game, in which product locations and prices are chosen by the market players. In equilibrium, locations will coincide whenever the strength of social interactions among consumers is grater than a critical value. By contrast, when the strength of social interactions is smaller than this critical value, one restores Hotelling's (1929) standard result according to which the distance between product variants is maximal.

Keywords: Heterogeneous preferences, product differentiation, social interactions, phase transition, discontinuity of demand, multiple market equilibria.

JEL Classification codes: D11, D4, M30

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1 Introduction

Becker (1991) proposed an interesting model to explain the demand behavior of two similar restaurants competing for consumers across from each other. He illustrates his model with the following interesting case: "... A popular seafood restaurant in Palo Alto, California, does not take reservations, and every day it has long queues for tables during prime hours. Almost directly across the street is another seafood restaurant with comparable food, slightly higher prices and similar service and other amenities. Yet this restaurant has many empty seats most of the time. Why doesn't the popular restaurant raise prices, which would reduce the queue for seats but expand profits?..."

In order to explain this puzzle (demand polarization and constancy in over-demanded restaurant prices), Becker supposes essentially that individual choices for restaurants are positively influenced by the restaurants' aggregated demands. Supposing such positive externalities in consumer choices, hereafter called *social interactions*, a slight increase in prices could not only eliminate the queue, but also cut an additional number of costumers who use to visit the restaurant just because it is permanently over-demanded. The resulting effect is that a slight increase in prices might reduce significantly (discontinuously) the restaurant's demand, bringing it below the restaurant's capacity and also below the profit maximizing sales level. Knowing this, the management of the over-demanded restaurant opt rationally not to raise its prices in spite of a positive excess demand.

In this paper we propose a model of heterogeneous interacting agents that supports and extends Becker's analysis. The model is inspired by Becker's duopoly model and the literature on phase transitions in social systems. Before presenting the novelties of our contribution, we present a brief literature review. A rich literature review until 2000 is provided by Glaeser and Scheinkman (2003).

Previous work. The occurrence of phase transition in social systems can be traced back to Föllmer (1974). In his pioneering paper applying statistical mechanics to economics, he modeled an exchange economy with two goods and many interacting consumers. Based on the Ising model, Föllmer shows the existence of two equilibrium price vectors in two respective phases of the economy.

Galam et al. (1982) proposed a model of strike that exhibits phase transition in the workers' level of production. Depending on the model parameters (the interaction strength among workers and their social permeability), the workers' level of production admits two (meta) stable states: a predominant strike state and a predominant work state. Hysteresis of production level is analyzed in respect to the workers' salary.

Gennotte and Leland (1990) shows that small supply shocks in stock markets may cause significant price falls (crashes) or high price volatility even if the extent of hedging in the market is not expressive. Crucial for the occurrence of price fall is that the proportion of uninformed investors (about the extent of hedging) is sufficiently large. In this model, mimetic contagion occurs as a consequence of poorly informed agents inferring stocks future prices from other agents' decisions. Bickhchandani et al. (1992) model and analyze this kind of mimetic contagion assuming complete rationality of agents. In this model, agents evaluate their actions based on their prior private information about the state of the world and the posterior information observed in the actions of others. Bickhchandani et al. (1992) shows that if agents decide in a sequential order, then the process of agents' decisions degenerates to a sequence of copies of decisions. Interestingly, this result is proved under the assumption that all agents are rational and fully informed that, up to a few number of initial decisions, all agents' decisions are merely copies of their predecessors' ones (information cascade).

Orlean (1995) proposed a similar model with the essential difference that the author drops the assumption of a sequential order in which decisions are made. In Orlean's model, agents have no objective information about the right proportion of independent agents in the population of agents, that is, agents that base their decisions on their own prior information about the state of the world. In Orlean's model, the resulting distribution of agents' decisions depends on the distribution of individual subjective guesses about the "right" fraction of independent agents in the population of agents.

Galam and Moscovici (1991) investigated formations of group consensus with analytic methods of Statistical Physics. A distinctive model result is that group consensus tend to polarize when group size decreases.

Becker (1991) explains how the market shares of two similar restaurants may be subject to phase transitions (and discontinuities in prices) when individual choices for restaurants are positively influenced by the popularity of the restaurants.

Kirman (1993) proposes a stochastic process to explain herd behavior as a model of recruitment for the exploitation of a common interest. The paper suggests a more general explanation of Becker's (1991) restaurant case: demand polarizations are explained as local maxima of the limit probability distribution of a stochastic process of decisions - not as a deterministic equilibrium.

Glance and Huberman (1993) examine the evolution of the fraction of effective cooperating individuals in producing a public good. In the proposed model, agents have the incentive to cooperate based on their expectations of inducing others to cooperate in the future. Individual expectations about the impact of one's attitude on the future behavior of others depend on the group size as well on other model parameters, as time horizon. Particularly interesting are investigations concerning group sizes that support cooperation.

Dalle (1997) studied the occurrence of phase transition in industry landscapes. He modeled an industry landscape as a set of firms adopting one of two available technologies. Due to positive externalities in technology choice, firms have the incentive to mimic one another's technology choice. This polarization tendency depends crucially on the industry under consideration: if the industry is such that the firms are well structured and possess their internal R & D, then the firms are more autonomous in their technology choice (high heterogeneity), whereas if the industry is such that the firms are typically small, not equipped with their own R & D, then the firms tend to copy the technology choice of others (low heterogeneity). Dalle analyzes the condition under which an industry landscape converges toward standardization or the coexistence of different technologies.

Challet and Zhang (1997), based on Arthur's (1994) El Farol Problem, propose a different approach for analyzing demand behavior in speculative markets. In the so-

called Minority Game, an odd number of agents chooses one of two possible actions and wins one point if the minority side is chosen. Based on their memory about the last observed minority and majority sides, agents "infer" the best action to choose and choose it accordingly. Of course, there is no objective best action to be inferred from the past (if there were one, it would be the worst one whenever the majority chooses it). In this game, the population of agents attains the highest level of cooperation when they split into nearly 50% at each side over time. The authors show that a high level of cooperation arises (about 50% at each side) when the size of the agents' memory reaches a critical size. Insightful experiments are performed in which the size of the agents' memory is changed as a consequence of an evolutionary process of the population of agents. Interestingly, the mean size of the agents' memory converges to the critical size, where cooperation predominates. For further related papers, see the Minority Game web page: www.unifr.ch/econophysics/minority/.

Nadal et al. (1998) model the choice process of a buyer that has to choose (at different times) a single seller to deal with. Long run fidelity to some sellers arises when the buyer's profit (supposed to be independent of sellers) exceeds a critical level. The authors emphasize a particular parametrization in which the corresponding invariant probability distribution of the choice process can be derived from a compromise between prospecting market information from dealing with different sellers and making immediate profit with the most profitable seller.

Yin (1998), and later Levy (2004), model outbreaks of collective adoption of attitudes from the point of view of threshold models. In these models each agent decides individually to join a certain attitude (for example, protest against the government) as soon as a fraction of the population, or more, does. Depending on the distribution of agents' specific threshold fractions, multiple equilibria of attitudes may occur or not. Yin (1998) shows how a change in the distribution of thresholds may be induced by external events, while Levy (2004) emphasizes the general forms of thresholds distribution that lead to multiple equilibria of attitudes.

In a more extensive work, Brock and Durlauf (2001a, 2001b) developed many results on discrete choice models in the presence of social interactions.

Weisbuch et al. (2001), and later Weisbuch and Staufer (2004), analyze the attitude contagion of economic agents (to buy or not to buy a certain product) through simulations of adapted models imported from Statistical Physics: Percolation model in Weisbuch et al. (2001) and Voter model in Weisbuch and Staufer (2004). In both works, the local conditional probabilities of the site states are coupled with a global observable of the system configuration. In both works, self organized criticality was observed and interpreted from a socioeconomic point of view.

Glaeser and Scheinkman (2003) provided a very good research review of interactions based models until 2000. Models are presented and discussed in a more general framework. The authors suggest three econometric approaches for measuring social interactions.

Gordon et al. (2005) and Nadal et al. (2005) studied the optimal strategy of a monopolist that operates in a market whose consumers are susceptible to one another's choices - to buy or not to buy the monopolist's product. In order to model the evolution of consumers' discrete choices, the authors proposed a stochastic Ising type model in which reservation prices of consumers depend on consumers peers' choices. They discuss the model properties in the case when consumers' idiosyncrasies are random but fixed and in the case when they are random and always updated over time. Assuming fixed consumers' idiosyncrasies, the authors show that the monopolist's strategy is subjected to the following phase transition: if the mean reservation price exceeds a critical price, or the monopolist's cost is lower than a critical cost, then the optimal monopolist's strategy jumps from a solution with a high price and a small number of buyers, to another one with a low price and a large number of buyers. A sufficient condition for this regime change is satisfied when social interactions among consumers is strong enough. Particulary interesting in this regime change is that it may occur even when the resulting monopolist's demand is decreasing and continuous - not necessarily multi-valued.

Present paper. Our contribution was originally inspired by Becker's duopoly model (1991) and is related to the literature cited in this text. Of particular reference to our paper are the works of Becker (1991), Brock and Durlauf (2001a, 2001b), Glaeser and Scheinkman (2003), Yin (1998), Levy (2004), Gordon et al. (2005) and Nadal et al. (2005). Especially, ideas about the way consumers are influenced by other consumers and by product prices is similarly proposed by Gordon et al. (2005) and Nadal et al. (2005).

We outline now our new results and relate them to previous literature. As mentioned before, we propose a model that supports and extends Becker's explanation about the demand behavior of two similar restaurants located across from each other. The model concerns a duopoly market of heterogeneous interacting agents choosing one or the other product variant. Through this model we show that demands for product variants behave discontinuously in prices whenever consumers are sufficiently homogeneous in their *intrinsic preferences* between the two product variants (such as the two restaurants in Becker's case). In this context, we define the *intrinsic preference* of a consumer as the difference in variants prices that would make the consumer be indifferent between the two product variants, supposed that both product variants have the same level of aggregate demand, i.e., supposed that both product variants are equally attractive from a social or informational³ point of view.

The model also shows that, if the dispersion of consumers' intrinsic preferences exceeds a critical value, then the resulting demand functions are well-behaved, that is, decreasing and continuous. This is also consistent with Grandmont's (1987) and Kirman's (1992) considerations about the role of agents' heterogeneity in economic models.

Interestingly enough, the high level of homogeneity in consumers' intrinsic preferences, necessary for the occurrence of phase transition of demand in our model, can be assured by a high level of similarity of product variants (similarity of the restaurants in Becker's description). In fact, when both product variants are highly similar – ultimately identical – a little price difference between them would lead a great majority of consumers to choose

 $^{^{3}}$ An agent may also choose the variant chosen by the majority because this choice may maximize the agent's posterior expected utility, given the information observed in the majority behavior (Bickhchandani et al. (1992) and Orlean (1995)).

the one with the lowest price, except in the case of social interactions in consumer's choices, which is excluded by the definition of consumers' intrinsic preferences. The rationale behind this is that nobody is willing to pay much more for good A than for good B when A and B are highly similar – this becomes obvious when A and B are identical. Consequently, when product variants are highly similar, all consumers' intrinsic preferences will be concentrated around zero. Thus, a high similarity of product variants implies a high homogeneity (concentration) of consumers' intrinsic preferences. Through the lens of our model, the above implication leads to the interesting conclusion that the asymmetry of demand, reported in Becker's restaurant case, occurs not in spite of products similarity, but it is in fact assured by it.

As another example of this, consider a duopoly consisting of two competing night clubs in a rural area where there are no other night life options. Suppose the night clubs differ from each other only by their spacial locations and prices. Assume also that consumers' houses are uniformly distributed over the rural area. Under these circumstances, the intrinsic preference of a consumer is the difference in prices between the two night clubs that would compensate for the difference in the distance between each night club and the consumer's house - supposing both night clubs are equally demanded, that is, equally attractive from a social point of view. In this example, it seems reasonable to assume that short distances between night clubs imply low dispersions of consumers' intrinsic preferences, and vice-versa. According to our model, demand phase transition occurs when the night clubs are located sufficiently close to each other; to the contrary, it is avoided when the night clubs are located relatively faraway from each other.

This finding opens interesting perspectives for constructing and studying Hotelling's (1929) type models under the additional supposition that consumers are susceptible to one another's choices. Due to the influence of product locations in consumers' intrinsic preferences, producers may locate their products and set their prices in order to produce or avoid a specific demand regime. Depending on the initial market share and the cost functions of producers, one or the other location-price strategy may be optimal. In this paper we propose such a location-price game and study its properties. We will still comment the game results in this introduction.

Another important novelty of the model is the consideration of two set of consumers: a time dependent set of demanding consumers and a larger, time independent set of all potential consumers from which the former is selected (quite arbitrarily and not necessarily at random). All potential consumers' intrinsic preferences are initially distributed according to a general probability distribution, which are supposed to be fixed over time. We also assume that only demanding consumers demand one or the other product variant at the respective demanding time. From these model definitions, we derive the stochastic process of the market shares of both product variants, hereafter *the market share process*. From the general structure of the market share process we deduce the following result (Proposition 1): when the number of demanding consumers goes to infinity, then the market share process converges to a well defined dynamical system, regardless of the particular selections of demanding consumers under consideration. This result suggests interesting socioeconomic implications and applications that we describe below. We investigate the market share process under the following selection of demanding consumers. The set of demanding consumers are always composed by a fraction of habitual consumers and a fraction of new consumers "refreshing the market" over time (model parameter between 0 and 1). According to the result stated in the previous paragraph (Proposition 1), the market share process will converge to the same dynamical system regardless of the fraction new consumers under consideration. Such a kind of result is discussed in Gordon et al. (2005) and Nadal et al. (2005) in the two particular cases when the fraction of new consumers "refreshing the market" is 0 (fixed heterogeneity over time) and when it is 1 (fluctuating heterogeneity over time).

Surprisingly enough, our limit result, that applies to any fraction of new consumers, allows us also to deduce interesting and insightful relationships between the behavior of the market share process and the fraction of new consumers being replaced over time in the case when the number of demanding consumers is finite - instead of infinite. Based on these relationships we explore the possibility of driving the distribution of attitudes of a social system into a desired distribution of attitudes. A concrete application of it concerns reduction of corruption in certain social systems. A little job-rotation among system members may lead to substantial reduction of chronic corruption among them.

Another contribution of our work is a price-game for the study of duopoly markets in the presence of social interactions of consumers. In the price-game, the initial market shares as well the product locations play an important role in determining the game equilibria. Product locations determine whether a demand phase transition will occur or not. This is because the occurrence of phase transition in our approach depends on the heterogeneity of consumers' intrinsic preferences, which in its turn is influenced by the products similarity - the products *proximity* as referred by Anderson et al. (1992) in the context of product locations in a characteristic space of products. The price-game shows that if product variants are sufficiently close to each other, then polarization of demand will occur. We notice that this result provides a complementary explanation to Becker's restaurant case by recalling that the restaurants under consideration were supposed to be similar and close to each other.

Moreover we propose a location-price game in which locations are part of the producers' strategy. An equilibrium analysis of this game shows that product locations coincide whenever the strength of social interactions among consumers is grater than a critical value. By contrast, if the strength of social interactions is smaller than the critical value, one restores Hotelling's (1929) standard result, according to which the distance between product variants is maximal.

In the next section we will define our formal model. Before doing so, we explain how the paper content is presented throughout the sections.

In Section 2 we present a formal model of heterogeneous interacting consumers deciding between two product variants.

In Section 3 we derive and study the market share process that results from the formal model defined in Section 2. In this section we first present Proposition 1. Proposition 1 shows that the trajectories of the market share process converge to the trajectories of a specific dynamical system when the number of consumers increases. Then, using Proposition 1, we analyze a model parametrization, according to which, a fraction of consumers (model parameter) are permanently replaced by new consumers "refreshing the population of consumers" over time. This gives rise to a kind of market share mix dynamics composed of two well known types of stochastic dynamics in Physics: annealed and quenched dynamics. We analyze such hybrid stochastic dynamics in the case of a finite number of demanding consumers and find two respective patterns for the market share process: an ergodic market share process,⁴ when the fraction of exchanging agents exceeds a critical fraction, and a non-ergodic one, in the opposite case. We discus an application of this process property and show how it can be used to drive the limit probability distribution of the market share process into a desired probability distribution.

In Section 4 we study the demand functions of product variants through the analysis of the dynamical system to which the market share process converges. In particular, we stress two regimes of demands: one in which the heterogeneity of consumers' preferences is lower than a critical level and the other in which it is higher than this critical level. Depending on whether low or high heterogeneity regime prevails, product demands will be subject to a phase transition or not. Here we also show that the discontinuity points of the market shares is determined by the initial market shares under consideration.

In Section 5 we show that the dispersion of consumers' intrinsic preferences is an increasing function of the distance between product variants. This result is presented in Proposition 2. Based on Proposition 2 and an introduced price-game, we show that short distances between product variants lead to polarization of demand. This is presented in Proposition 3. In this proposition (Proposition 3) we show also that large distances between product variants lead to the opposite result, namely, the market is symmetrically shared among competitors. We also investigate a generalization of the price-game, the location-price game, according to which players first locate their products and then play the price-game mentioned before. An equilibrium analysis of the location-price game shows that market players locate their products close to each other if the strength of social interactions among consumers is grater than a critical value. To the contrary, players locate products faraway from each other if the strength of social interactions among consumers is smaller than this critical value. This will be the content of Proposition 4.

In Section 6 we present our final remarks. In Section 7 we provide an appendix containing technical proofs of Proposition 1, Proposition 2 and Proposition 3.

2 The model

In this section we will present a duopoly model where consumers are susceptible to one another's decisions. Although our model covers several socioeconomic systems, where agents face a binary choice, we focus on the exemple of two competing establishments denoted by $E^{(-)}$ and $E^{(+)}$, say, two fashionable night clubs or restaurants, charging entrance prices $p_{(-)}$ and $p_{(+)}$, respectively.

 $^{{}^{4}}$ By an ergodic market share process we mean a stochastic market share process whose limit probability distribution does depend on the process initial condition.

We assume that at each time t = 0, 1, 2, ..., say, weekends, N consumers, selected from a set C of all potential consumers, $|C| \ge N$,⁵ decide between two establishments.⁶ We will denote by $C_t^{(N)}$ the set of N consumers selected at time t. For example, if |C| =100, N = 3 and consumers 5, 20 and 99 are selected at time t = 1 then $C_1^{(3)} = \{5; 20; 99\}$. The selection of $C_t^{(N)}$ may be quite general, but the *intrinsic preferences* of consumers belonging to $C_t^{(N)}$ must be stochastically independent from each other (this should hold for each time $t \in \{1, 2, 3...\}$). This independence will be formalized later in (6), after we have defined the intrinsic preferences of consumers.

Let us denote the number of consumers that choose establishment $E^{(-)}$ and $E^{(+)}$ at time t by $N_t^{(-)}$ and $N_t^{(+)}$ respectively. Let us also assume that all potential consumers of both establishments (all elements of C) are aware of the popularity of both establishments in the recent past. In order to reflect this in our model, we assume that at each time t > 0, each consumer $i \in C$ knows the fractions $N_{t-1}^{(-)}/N$ and $N_{t-1}^{(+)}/N$, where $N_{t-1}^{(-)}/N = \bar{N}_0^{(-)}$ and $N_{t-1}^{(+)}/N = \bar{N}_0^{(+)}$ (the initial market shares, $N_0^{(-)}$ and $\bar{N}_0^{(+)}$, are two model parameters that do not vary with N).

Let us now present the preference structure of consumers. For $t \ge 1$ and $i \in C_t^{(N)}$, let us denote by $U_t^{(i)}(x)$ the utility of the consumer *i* in deciding for $E^{(x)}$, $x \in \{-,+\}$ at time *t*, and queueing if necessary. To focus our attention on the main phenomena we are going to explain, we propose the following utility functions:

$$U_t^{(i)}(x) \stackrel{\text{\tiny def}}{=} J \frac{N_{t-1}^{(x)}}{N} - p(x) + u^{(i)}(x),$$

$$\forall t \in \{1, 2, \ldots\}, \ \forall i \in C_t^{(N)}, \ \forall x \in \{-, +\},$$
(1)

where J, p(x) and $u_t^{(i)}$ are explained below.

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- J is a positive parameter that measures the level of social influence on the utilities of consumers. This interpretation of J is clear from the fact that $N_{t-1}^{(x)}/N$ expresses the popularity of establishment $E^{(x)}$ in the immediate past($x \in \{-, +\}, t > 0$). The positivity of J follows Becker's (1991) explanation about positive externalities in consumers' choices. As stated by him "...The motivation for this approach is the recognition that restaurant eating, watching a game or play, attending a concert,... are all social activities in which people consume a product or service together and partly in public..."
- p(x) is positive and denotes the price charged by establishment $E^{(x)}$ $(x \in \{-,+\})$.
- $u^{(i)}(x)$ may be positive or negative and denotes a consumer's *i* specific utility increment in choosing establishment $E^{(x)}$ at time *t*. The quantity $u^{(i)}(x)$ can also be interpreted as the negative value of transport cost incurred by consumer *i* by visiting establishment $E^{(x)}$ $(i \in C_t^{(N)}, x \in \{-,+\}, t \ge 0)$.

 $^{{}^{5}|}C|$ denotes the number of elements of C.

⁶We suppose that the fraction of potential consumers that indeed decide between $E^{(-)}$ and $E^{(+)}$ (instead of neither of them) is more or less constant over time. Accordingly, N does not depend on t.

For $t \ge 1$ and $i \in C_t^{(N)}$, let us denote by $x_t^{(i)} \in \{-,+\}$ the decision of consumer *i* at time *t*; that is, if $x_t^{(i)} = x$, then the consumer *i* goes to establishment $E^{(x)}$ ($x \in \{-,+\}$). The utility maximization behavior implies that

$$U_t^{(i)}(x_t^{(i)}) = \max_{x \in \{-,+\}} U_t^{(i)}(x)$$
(2)

Decision $x_t^{(i)}$ is uniquely determined by equation (2) when $U_t^{(i)}(-) \neq U_t^{(i)}(+)$. Let us assume $x_t^{(i)} = +$ when $U_t^{(i)}(-) = U_t^{(i)}(+)$. As we will see, this assumption does not cause an asymmetry in the resulting aggregate demand, since in accordance with further descriptions of the model, the event $\{U_t^{(i)}(+) - U_t^{(i)}(-) = 0 \text{ for some } i \in C_t^{(N)}\}$ occurs with probability zero.

Now, from the utility functions defined in (1) and utility maximization (2) it follows for $t \ge 1$ and $i \in C_t^{(N)}$:

$$x_t^{(i)} = \begin{cases} +1, & \text{if} & p(+) - p(-) \leq \theta^{(i)} + Jm_{t-1}^{(N)} \\ -1, & \text{otherwise} \end{cases}$$
(3)

where $m_{t-1}^{(N)} \stackrel{\text{\tiny def}}{=} [N_{t-1}^{(+)} - N_{t-1}^{(-)}]/N$ and $\theta^{(i)} \stackrel{\text{\tiny def}}{=} u^{(i)}(+) - u^{(i)}(-).$

The quantity $\theta^{(i)}$ can be interpreted as a consumer's *i* reservation price difference for $E^{(+)}$, free from bias in social influence. That is, $\theta^{(i)}$ is the maximal price difference p(+) - p(-) consumer *i* is willing to pay in order to acquire a unit of $E^{(+)}$ instead of $E^{(-)}$, supposing there is no bias in social influence, i.e., supposing $m_{t-1}^{(N)} = 0$. The value of θ_i , which may be negative, zero, or positive, reveals an intrinsic preference of consumer *i* for product variants $E^{(-)}$ and $E^{(+)}$.

In accordance with the above,

 $\theta^{(i)}$ will be called *intrinsic preference* of consumer $i \ (i \in C)$

Note that $\theta^{(i)}$ is time independent and is defined for each potential consumer $i \in C$. We anticipate that the dispersion of consumers' intrinsic preferences, $\theta^{(i)}$, $i \in C$, will play a crucial role in determining if demands for establishments are continuous functions of their prices or not.

2.1 Heterogeneity of consumers' preferences

In the previous model definitions we introduced the consumers' intrinsic preferences as being consumers' specific characteristics. Although these characteristics may differ among consumers, we propose a certain statistical pattern for the ensemble of these characteristics, which should be typical for the population of potential consumers under study.

In order to model a pattern in the variability of consumers' preferences, we assume that the consumers' intrinsic preferences $\theta^{(i)}$, $i \in C$, are independent and identically distributed

random variables, each composed of a mean intrinsic preference $\overline{\theta}$ and a consumer's *i* specific deviation $-\xi^{(i)}$ from this mean $(i \in C)$, i.e.,

$$\theta^{(i)} = \overline{\theta} - \xi^{(i)}, \quad i \in C \tag{4}$$

where $-\xi^{(i)}$, $i \in C$, are independent and identically distributed random variables, which may assume positive or negative values. The negative sign of $-\xi^{(i)}$ is due to convenience in further exposition.

We assume that the random variables $\xi^{(i)}$, $i \in C$ (without minus sign), have a cumulative distribution function Φ whose derivative Φ' satisfies the following properties:

1) Unimodality: Φ' is increasing in $(-\infty, 0]$ and decreasing in $[0, \infty)$ 2) Symmetry: $\Phi'(z) = \Phi'(|z|) \quad \forall z \in \mathbb{R}$ (5)

The symmetry of Φ' implies that $\overline{\theta} = \mathbb{E}(\theta^{(i)}), i \in C$, where $\mathbb{E}(\theta^{(i)})$ denotes the expected value of the random variable $\theta^{(i)}$. This fact justifies the name "mean intrinsic preference" given to $\overline{\theta}$. In our model, $\overline{\theta}$ is a time independent parameter. It allows us to model exogenous intervention like promotions, advertising, etc., that may shift the mean intrinsic preference of consumers without affecting the deviations $\xi^{(i)}, i \in C$, from this mean.

With respect to assumptions (5), we note the following: i) we expect that a unimodal distribution of ξ_i , $i \in C$, would be a good first order approximation of a real distribution; ii) the symmetry of Φ' is assumed in order to simplify our explanation. As we will see, analogous results to those that we will present can be derived under even milder conditions than (5).

2.2 Selection of consumers

As described before, our model assumes that at each time t, a set $C_t^{(N)}$ of "selected consumers" choose between the two establishments, either $E^{(-)}$ or $E^{(+)}$. Consumers from $C - C_t^{(N)}$ choose neither of them at time t. In this section we formalize and discuss the precondition imposed on the selection of consumer sets $C_t^{(N)}$, t = 1, 2, ...

First of all, we notice that every restriction imposed on the selection of $C_1^{(N)}, C_2^{(N)}, \ldots$, imposes ultimately a restriction on (time-dependent) consumers' preferences. To see this, note that each consumer *i* faces, in fact, three choices: either 1) $E^{(+)}$, 2) $E^{(-)}$ or 3) nither of them. Thus, a consumer *i*, who does not belong to $C_t^{(N)}$, prefers 3) instead of 1) or 2) at time *t*. As we will see, we do not impose many restrictions on the time dependent preferences of those consumers (who prefer to go somewhere else rather than $E^{(+)}$ or $E^{(-)}$ at time *t*). This is in fact the reason why we model the set of consumers who prefer at least one of both $E^{(+)}$ or $E^{(-)}$ to neither of them at time *t* with a generic set $C_t^{(N)}$ $(t = 1, 2, \ldots)$.

The only precondition we imposed on the set of selected consumers $C_t^{(N)}$ is that the intrinsic preferences of the consumers belonging to $C_t^{(N)}$ should be stochastically independent from one another. Since $\theta^{(i)} = \bar{\theta} + \xi^{(i)}$ and since $\bar{\theta}$ is a constant, this independence

can be formalized as follows. For each (fixed) time $t \in \{0, 1, 2, \ldots\}$,

$$\xi^{(i)}, i \in C_t^{(N)}$$
 are independent random variables (6)

Note that the independence above does not imply that $\xi^{(i)}, i \in C_s^{(N)}$ and $\xi^{(i)}, i \in C_t^{(N)}$ are independent random variables for $s \neq t$. In fact this independence does not hold when $C_s^{(N)}$ and $C_t^{(N)}$ intercept.

We observe that condition (6) is satisfied when the selection of $C_t^{(N)}$ is independent on the configuration of intrinsic preferences $\theta^{(i)}, i \in C$. Since $\theta^{(i)} = \overline{\theta} + \xi^{(i)}$, this independence could also be formalized as follows: for any time t and any subset \mathcal{C} of N elements of $C = \{1, 2, \ldots, |C|\}$ $(N \leq |C|)$, it holds that

$$\mathbb{P}\left(C_t^{(N)} = \mathcal{C} \left| \xi^{(1)}, \dots, \xi^{|C|} \right) = \mathbb{P}\left(C_t^{(N)} = \mathcal{C}\right)$$
(7)

Condition (7) states that the probability of selecting a particular subset C from the set of all potential consumers C at time t does not depend on the configuration of consumers' intrinsic preferences, solely determined by $(\xi^{(1)}, \xi^{(1)}, \dots, \xi^{(|C|)})$.

Roughly speaking, condition (7) imposes the following restriction on consumers' preferences: consumers' (random) preference orders restricted to the two choices, i) " $E^{(-)}$ or $E^{(+)}$ " and ii) "neither of them", should be independent of consumers' (random) preference orders restricted to the two choices, i) $E^{(-)}$ and ii) $E^{(+)}$. This implies, for example, that the probability of event " $i \in C_t^{(N)}$ " does not change if we know that i is more likely to prefer one specific side, say $E^{(+)}$, that is, if we know that $\xi^{(i)}$ is small (see (3) and (4)).

In Section 3 we will use condition (6), which follows from (7), to deduce some interesting properties for the emerging demand behavior of $E^{(+)}$ and $E^{(-)}$. In particular, the formulation of the sets of consumers who choose either $E^{(+)}$ or $E^{(-)}$ at time t = 1, 2, ... in terms of generic sets $C_1^{(N)}, C_2^{(N)}, ...$, will enable us to deduce different demand behaviors depending on different structures of consumers' preferences. These structures could be of the following three types: i) $C_t^{(N)} = C_0^{(N)}$, for all t = 1, 2, ..., in this case, consumers' preferences do not change over time; ii) $C_s^{(N)} \cap C_t^{(N)} = \emptyset$, for $s \neq t$, in this case, consumers' preferences change completely over time; and iii) $C_s^{(N)}$ and $C_t^{(N)}$ overlap partially for some $s \neq t$, in this case, consumers' preferences change partially over time.

We will come back to this point in the next section.

3 The market share process

In this section we will present a description of the market share process, that is, the twodimensional stochastic process $(N_t^{(-)}/N, N_t^{(+)}/N)$. Recall that $N_t^{(-)}/N$ and $N_t^{(+)}/N$ are the respective market shares of $E^{(-)}$ and $E^{(+)}$ at time t. We will also analyze the asymptotical behavior of the market share process when N (number of consumers) increases.

In order to allow us an easy treatment of the market share processes, let us describe it through the equivalent stochastic process of *difference of demand fractions*:

$$m_t^{(N)} = (N_t^{(+)} - N_t^{(-)})/N$$
(8)

That $(m_t^{(N)})$ is equivalent to the two-dimensional market share process $(N_t^{(-)}/N, N_t^{(+)}/N)$, is due to the fact that: $N_t(+1)/N + N_t(-1))/N = 1$, $\forall t \ge 0$. In fact, the latter relation implies that the value of $m_t^{(N)}$ follows from the values of $N_t^{(-)}/N$ and $N_t^{(+)}/N$ and viceversa.

Now, by construction, the difference of demand fractions equals the arithmetic mean of decisions:

$$m_t^{(N)} = \frac{1}{N} \sum_{i \in C_t^{(N)}} x_t^{(i)}$$
(9)

Relationships (3), (4) and (9) altogether allow us to present stochastic process $(m_t^{(N)})$ in the following way. First of all, generate |C| independent random variables $\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(|C|)}$ with distribution function Φ ; set $m_0 \stackrel{\text{def}}{=} \bar{N}_0^{(+)} - \bar{N}_0^{(-)7}$. For t > 0, determined $m_t^{(N)}$ by the following steps:

- 1. Choose set $C_t^{(N)}$, of N consumers from the population $C = \{1, 2, ..., |C|\}$ in accordance with the specified selection rule.
- 2. Use the values of $m_{t-1}^{(N)}$ and $\xi^{(i)}, i \in C_t^{(N)}$, to define

$$\begin{aligned} x_t^{(i)} &= \begin{cases} +1, & \text{if} \\ -1, & \text{otherwise} \end{cases} \quad \xi^{(i)} \leq Jm_{t-1}^{(N)} - h \\ \text{,} \quad \forall i \in C_t^{(N)} \end{aligned}$$

where $h \stackrel{\text{\tiny def}}{=} p(+1) - p(-1) - \bar{\theta}$.
Set $m_t^{(N)} = \frac{\sum_{i \in C_t^{(N)}} x_t^{(i)}}{N}$

Large population dynamics. Steps 1), 2) and 3) above describe a family of stochastic processes. Each stochastic process belonging to this family is determined by the sequence of sets $C_t^{(N)} t \ge 1$ under consideration. Although we do not specify how sets $C_1^{(N)}, C_2^{(N)}, \ldots$ are selected (the selection rule of consumers), the asymptotical behavior of these processes will be the same: all stochastic processes belonging to this family will converge to the following dynamical system (when the number of consumers N increases)

$$m_t = 2\Phi(Jm_{t-1} - h) - 1, \quad t = 1, 2, \dots$$
 (10)

where we recall that Φ is the cumulative probability function of ξ_1 .

The next proposition formalizes this result.

3.

Proposition 1. Let $\xi^{(i)}$, i = 1, 2, ... be identically distributed random variables defined on a common probability space with a common continuous cumulative distribution function Φ . Let $C_t^{(N)}$ be a sequence of sets of positive integers (indexed by t and N) where $|C_t^{(N)}| = N$

 $^{{}^{7}\}bar{N}_{0}^{(-)}$ and $\bar{N}_{0}^{(+)}$ are parameters which do not depend on N.

for all $N \ge 1$, $t \ge 1$. Assume that, for each $N \ge 1$, $t \ge 1$: $\xi^{(i)}$, $i \in C_t^{(N)}$ are independent random variables. If for all $N \ge 1$, $m_0^{(N)} = m_0$, and for t > 0, $m_t^{(N)}$ is recursively defined by the steps 1)-3) of Section 3, then

$$\forall t \ge 0: \quad \lim_{N \to \infty} \left| m_t^{(N)} - m_t \right| = 0 \quad almost \; surely \;, \tag{11}$$

where m_t denotes the t-th iteration of mapping $m \mapsto 2\Phi(Jm-h)-1$, starting from m_0 , that is, $m_t = 2\Phi(Jm_{t-1}-h)-1$ for t > 0.

The proof of Proposition 1 is presented in the Appendix.

Proposition 1 states that the trajectories of stochastic process $(m_t^{(N)})$ converge pointwise (that is, for each t = 0, 1, 2...) almost surely to the trajectories of dynamical system (m_t) when the number of consumers goes to infinity.

An important part of this paper is devoted to the study of dynamical system (m_t) that is a proxy for the large population dynamics of $(m_t^{(N)})$. However, before we proceed with this study, it is worth analyzing the behavior of stochastic process $(m_t^{(N)})$ when N, the number of consumers, is finite.

Exchanging agents. Although Proposition 1 formalizes the behavior of process $(m_t^{(N)})$ when the number of consumers goes to infinity, its degree of generality (valid for a large class of sequences of sets $C_t^{(N)}$, $t \ge 1$, $N \ge 1$), enables us to derive some interesting conclusions about $(m_t^{(N)})$ in the case when N is finite.

In order to present these conclusions let us recall that the random variables $\xi^{(i)}$, $i \in C$ do not change their values over time.

Although $\xi^{(i)}$, $i \in C$, do not change over time, the amount of effective fixed and effective varying $\xi^{(i)}$'s in the system will depend on the sequence of sets $C_1^{(N)}, C_2^{(N)}, \ldots$ If, for example, the sets of consumers who decides either for $E^{(+)}$ or $E^{(-)}$ are always the same $(C_t^{(N)} = C_0^{(N)}, \forall t)$, then we can assume that consumers' preferences do not change over time. This corresponds in the literature to the random utility models of MacFaden (1974) and Manski (1977). On the other hand, if sets $C_1^{(N)}, C_2^{(N)}, \ldots$ change completely over time $(C_t^{(N)} \cap C_s^{(N)} = \emptyset, \forall s \neq t)$, then the process behaves as if agents were fixed, but with varying preferences over time. This corresponds in the literature to Thurstone's (1927) random utility model.

Gordon et al. (2005) and Nadal et al. (2005) show the correspondence of these two particular cases (fixed preferences and changing preferences over time) to two respective types of models in Statistical Physics: models with quenched disorder and models with annealed disorder. As mentioned before, these two types of models correspond in our model to two particular sequences of selected sets of agents. Besides these two particular cases, there are other possibilities covered by our model.

This fact and its applications in social systems make the discussion of the following selection scheme important.

Let $C = \{1, 2, 3...\}$ and for each $t \ge 0$, set

$$C_t^{(N)} = \left[\left\{ 1, 2, \dots, N \right\} - Old_t^{(\alpha N)} \right] \cup New_t^{(\alpha N)}$$
(12)

where $Old_t^{(\alpha N)}$ is a set of $[\alpha N]$ elements, $0 \le \alpha \le 1$, chosen at random from $\{1, 2, \ldots, N\}$, and $New_t^{(\alpha N)}$ is a set of $[\alpha N]$ elements selected from $\{N+1, N+2, \ldots\}$ with $New_s^{(\alpha N)} \cap New_t^{(\alpha N)} = \emptyset$ for $s \ne t$. Above $[\alpha N] \stackrel{def}{=} Min\{k \in \mathbb{Z} : \alpha N \le k\}$.

In definition (12) we also assume that the procedure of selection of $New_t^{(\alpha N)}$, t = 0, 1, 2... does not depend on $\xi_{(i)}, i \in C$. We do not specify this procedure because it is not relevant for the stochastic update of $(m_t^{(N)})$.⁸

Now consider the following three ranges of values of α (fraction of new agents):

- When $\alpha = 0$, then $C_t^{(N)} = \{1, 2, ..., N\}$ for all t = 0, 1, 2... In this case there is no randomness over time; that is, all $m_t^{(N)}$, t = 1, 2, ... are solely determined by the initial value of m_0 and the (fixed) values of $\xi^{(1)}, \xi^{(2)}, ..., \xi^{(N)}$. Here, agents' preferences are fixed.
- When $\alpha = 1$, then $C_s^{(N)} \cap C_t^{(N)} = \emptyset$ for $s \neq t$. In this case the consumers visiting the establishments have never visited them before. Process $(m_t^{(N)})$ has a stochastic time update and is Markovian. Here, agents' preferences change completely over time.
- When $0 < \alpha < 1$, then the establishments are visited by a positive fraction α of new consumers and a positive fraction 1α of old consumers. Here, agents' preferences change partially over time. Of the three cases, this case comes the closest to real life.

Our interest in (12) focuses on a relationship between the ergodicity of stochastic process $(m_t^{(N)})$ and a range of values of α . Under some conditions it will be shown that $(m_t^{(N)})$ is ergodic⁹ if and only if α is grater than a critical value α_* .

In order to discus this relationship, we first observe that stochastic process $(m_t^{(N)})$ converges to dynamical system (m_t) for all values of $\alpha \in [0, 1]$. This convergence result follows immediately from Proposition 1, since according to (12), variables $\xi^{(i)}$, $i \in C_t^{(N)}$ are independent random variables.

Based on convergence $(m_t^{(N)}) \to (m_t)$, which holds for any value of $\alpha \in [0, 1]$, we will discuss two implications. To present these implications we consider the following representation of stochastic process $(m_t^{(N)})$:¹⁰

$$m_t^{(N)} = 2\Phi_N^{(t)} (Jm_{t-1}^{(N)} - h) - 1$$
(13)

⁸The transition probabilities of stochastic process $(m_t^{(N)})$ do not depend on the specific procedure of selection of $New_t^{(\alpha N)}$, provided that this procedure does not depend on $\xi^{(i)}$, $i \in C$. One possible selection scheme is: $New_t^{(\alpha N)} = \{(N + t[\alpha N]) + 1, (N + t[\alpha N]) + 2, \dots, (N + t[\alpha N]) + [\alpha N]\}, t = 0, 1, 2 \dots$

⁹We say that $(m_t^{(N)})$ is ergodic if the limit probability distribution of $m_t^{(N)}$ (when $t \to \infty$) does not depend on the initial value $m_0^{(N)}$.

¹⁰That (13) holds, follows directly from the equalities $m_t^{(N)} = (1/N) \sum_{i \in C_t^{(N)}} x_t^{(i)} = (N_t^{(+)} - N_t^{(-)})/N = 2[N_t^{(+)}/N] - 1$ and from the fact that $x_t^{(i)} = 1$ if, and only if $\xi^{(i)} < Jm_{t-1}^{(N)} - h$ (see step 2 of the update description of $(m_t^{(N)})$).



Figure 1: Convergence of sequence $(m_t^{(N)})_{t\geq 0}$ (starting from $m_0^{(N)} = m_0$) to a fix point of mapping $m \to g_N(m) \stackrel{\text{def}}{=} 2\Phi_N^{(0)}(Jm-h) - 1$ (step function). This fix point of $m \to g_N(m)$ is close to M_+ , where M_+ is a fix point of $m \to g(m) \stackrel{\text{def}}{=} 2\Phi(Jm-h) - 1$ (continuous function).

where $\Phi_N^{(t)}$ is the empirical distribution of random variables $\xi^{(i)}$, $i \in C_t^{(N)}$, that is

$$\Phi_N^{(t)}(x) \stackrel{\text{\tiny def}}{=} \frac{1}{N} \sum_{i \in C_t^{(N)}} \mathbb{I}_{\{\xi^{(i)} < x\}}, \text{ where } \mathbb{I}_{\{\xi^{(i)} < x\}} \stackrel{\text{\tiny def}}{=} \begin{cases} 1 & \text{if } \xi^{(i)} < x \\ 0 & \text{otherwise} \end{cases}$$

We also assume that dynamical system (m_t) $(m_t = 2\Phi(Jm_{t-1} - h) - 1)$ has two stable equilibria, M_+ and M_- (see Figure 1).¹¹ Taking into account that, for each fixed t, the sequence of functions $\{x \mapsto \Phi_N^{(t)}(x) : N = 1, 2, ...\}$ converges uniformly, almost surely to the cumulative distribution function $x \mapsto \Phi(x)$ (see (38) in Proof of Proposition 1 in the Appendix), we have the following implications.

If the number of agents N is large (not necessarily infinite) and $\alpha = 0$, then $(m_t^{(N)})$ has no randomness over time, and $(m_t^{(N)})$ is attracted to a neighborhood either of M_+ or M_- where it remains locked forever. In fact, if N is large, then the graph of $\Phi_N^{(0)}$ is close to the graph of Φ , and if $\alpha = 0$, then $C_t^{(N)} = C_0^{(N)}$, $t = 1, 2, \ldots$, that is $\Phi_N^{(t)} = \Phi_N^{(0)} \sim \Phi$, $t = 1, 2, \ldots$ accordingly $m_{t+1}^{(N)} = 2\Phi_N^{(0)}(Jm_t^{(N)} - h) - 1$, and $(m_t^{(N)})$ converges to a fix point of $m \mapsto 2\Phi_N^{(0)}(Jm - h) - 1$, which is close either to M_+ or M_- (see Figure 1). If we now allow α to be a little larger than zero, but still relatively small, then just a few $\xi^{(i)}$'s

¹¹The existence of two stable equilibria, M_+ and M_- , can be deduced from a low dispersion of agents' preferences, that is, low dispersion $\xi^{(i)}$'s.

(agents' preferences) will be randomly replaced over time. Under this new value of α , it is natural to suppose that after a certain relaxing time, say t_* , the values of $m_{t_*}^{(N)}, m_{t_*+1}^{(N)}, \ldots$ will also remain locked within a neighborhood either of M_+ or M_- forever despite process randomness over time.

Now, if N is finite and $\alpha = 1$ ($\alpha = 1$ means that all agents' preferences are replaced over time), then $(m_t^{(N)})$ has only one limit probability distribution (when $t \to \infty$). If we also suppose that N is relatively large (but still finite), then the only limit probability distribution of $(m_t^{(N)})$ is close to a bimodal probability distribution, where the modes are close to M_- and M_+ respectively. As observed by Kirman (1993), in such a case there is no deterministic equilibrium of $(m_t^{(N)})$ (when $t \to \infty$), but a limit probability distribution of $(m_t^{(N)})$ that is bimodal.

Let us summarize the two implications:

- 1. If N is relatively large (not necessarily infinite) and α is sufficiently small (not necessarily zero), then $(m_t^{(N)})$ is attracted to a neighborhood either of M_- or of M_+ where it remains locked forever. The limit probability distribution of $(m_t^{(N)})$ depends on the initial value of $m_0^{(N)}$. In this case, $(m_t^{(N)})$ is not ergodic, given the random parameters $\xi^{(1)}, \xi^{(2)}, \ldots \xi^{(N)}$.
- 2. If N is finite, but relatively large, and $\alpha = 1$, then $(m_t^{(N)})$ has a unique limit distribution that does not depend on the initial value of $m_0^{(N)}$. The limit probability distribution of $(m_t^{(N)})$ is bimodal and the two modes are close to M_- and M_+ respectively. In this case, $(m_t^{(N)})$ is ergodic.

The above discussion leads to the following conjecture: if N is large enough and dynamical system (m_t) has more than one stable equilibrium, then there will be a critical fraction $\alpha_*^{(N)} \in [0,1]$ such that the stochastic process $(m_t^{(N)})$, conditioned on random variables $\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(N)}$, is not ergodic if $\alpha \leq \alpha_*^{(N)}$ and ergodic if $\alpha > \alpha_*^{(N)}$. Although the critical fraction $\alpha_*^{(N)}$ is random (it depends on random variables $\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(N)}$), it is easy to see that $\alpha_*^{(N)}$ will converge to a fraction $\alpha_*(\Phi)$ that depends only on distribution function Φ and parameters J and h. Although $\alpha_*(\Phi)$ depends on parameters J and htoo, we can incorporate these parameters in the distribution function Φ . In this case, Φ would denote the distribution function of $(\xi_{(1)} + h)/J$, instead of $\xi_{(1)}$ (see step 2 of the time update of $(m_t^{(N)})$).

One suggestive application of the existence of $\alpha_*(\Phi)$ is related to the possibility of moving the fraction α , above and below $\alpha_*(\Phi)$ at different times, in order to set $(m_t^{(N)})$ free to scape from an undesired region and to lock it in a desired one. Suppose for example, that a governmental institution, divided into many departments, exhibits a high level of corruption among their members. Each member opts either to be corrupt (-) or not to be corrupt (+) according to social influences among department members¹² and the corresponding prices for the two possible attitudes: p(-) = risk of being caught and

 $^{^{12}}$ We assume that social interactions occurs among members of the same department, only.

punished, not to mention losses in moral integrity, and p(+) = the amount of money that will not be cashed. A possible way to move the institution away from a high level of corruption to a low level of corruption involves not only regulations on prices p(-), p(+), but also regulations on a fraction of members being exchanged across departments, that is, job rotation across departments. In this context, each department could be viewed as a separate market offering two products,¹³ where the fraction of job rotations across departments would work asymptotically as a fraction of new agents being exchanged in all markets. An efficient¹⁴ fraction of job-rotation across departments (about $\alpha_*(\Phi)$) could be implemented in order to induce phase transitions in all departments. Once this passage is made, one could stop job rotation ($\alpha \downarrow 0$) in order to lock the process in the desired state of low level of corruption.

The above discussion motivates several directions of research. First, one could investigate the dependence between $\alpha_*(\Phi)$ and Φ . Second, one could study the probability distribution of time T process $(m_t^{(N)})$ spends on its two meta-stable states (from which the respective expected values are close to M_- and M_+). A natural supposition is that the expected time $E_{\alpha}(T)$ is a decreasing function of the fraction of exchanging agents α , where $E_{\alpha}(T) \uparrow \infty$ when $\alpha \downarrow \alpha_*(\Phi)$. It would be interesting to investigate how fast $E_{\alpha}(T)$ decreases when α increases in $(\alpha_*(\Phi), 1]$. Investigating this would be important as long as we want to control the expected waiting time for the occurrence of phase transition in the system.

4 Demand equilibria

In the preceding section we described how the market shares of both establishments evolve in time. This was expressed by stochastic process $(m_t^{(N)})$ and by the dynamical system (m_t) that approaches $(m_t^{(N)})$ in the case where the number of consumers N is large.

In this section we proceed with the study of the equilibria of dynamical systems (m_t) as well as the economic interpretation of it. To start with, recall that m_t is the t-th interaction of $m \mapsto 2\Phi(Jm-h) - 1$, that is

$$m_t = 2\Phi(Jm_{t-1} - h) - 1, \quad t = 1, 2, \dots$$
 (14)

where Φ is the cumulative probability distribution function of ξ_1 .

Under the supposition that Φ satisfies (5), the graph of $m \mapsto 2\Phi(Jm-h) - 1$ has an *S*-shape as indicated in Figure 2. In Figure 2 the horizontal axis depicts *m* and the curves represent the graph of $m \mapsto 2\Phi(Jm-h) - 1$ for different values of parameters $\Phi'(0)^{-1}$, *J* and *h*. Below we comment the Curves *a*) - *f*):

 $^{^{13}}E^{(-)}$ = additional income (due to corruption) × $E^{(+)}$ = job without risk (due to a honest attitude). 14 An excessive fraction of job rotation could damage the institution functionality.

- a) This curve correspond to case where $\Phi'(0)^{-1} \ge 2J$. In this case, (m_t) has just one global stable equilibrium $\overline{M}(h)$. Increasing h translates the curve $m \mapsto 2\Phi(Jm - h) - 1$ from the left to the right, and the global equilibrium $\overline{M}(h)$ decreases.
- b) f) These last five curves correspond to cases where $\Phi'(0)^{-1} < 2J$. Each curve relates to a specific range of values of h. As before, increasing h translates the curve $m \mapsto 2\Phi(Jm - h) - 1$ from the left to the right as shown in b) - f). Depending on the value of h, dynamical system (m_t) has at least one and at most three equilibria $M_-(h)$, M(h) and $M_+(h)$, where $M_-(h)$ and $M_+(h)$ are stable equilibria and M(h) is an unstable equilibrium. Curves c) and e correspond to special values of h, namely $h = -h_*$ and $h = h_*$ respectively. The value of h_* is uniquely determined by the following equation system - to be solved in $(h_*, M_*) \in [0, \infty)^2$: $M_* = 2\Phi(JM_* - h_*) - 1$, $1 = 2\Phi'(JM_* - h_*)J$.

We stress the fact that the equilibria mentioned above depend on parameter h. From the relationships explained in Figure 2 we can deduce the functions that assign the values of h to the equilibria of dynamical system (m_t) . Figure 3 shows the unique global stable equilibrium $\overline{M}(h)$ of (m_t) as a function of h in the case $\Phi'(0)^{-1} \geq 2J$. Figure 4 shows the equilibria $M_-(h)$, M(h) and $M_+(h)$ of (m_t) as a function of h in the case $\Phi'(0)^{-1} < 2J$.

In what follows we will discuss and interpret the stability properties of equilibria $\overline{M}(h)$, $M_{-}(h)$, M(h) and $M_{+}(h)$. For this discussion we assumed in (5) that Φ' is symmetric around zero. The symmetry of Φ' ensures the existence of two regimes: i) $\Phi'(0)^{-1} \ge 2J$, where only one global stable equilibrium exists and ii) $\Phi'(0)^{-1} < 2J$, where multiple equilibria may exist for the same value of h. At this point it is worth mentioning that our results also apply if we relax the symmetry assumption imposed on Φ' . If for example, Φ' is no longer symmetric around zero, but still has only one pick at zero, then the graphs of mapping $m \mapsto 2\Phi(Jm - h) - 1$ would also display S-shapes as those of Figure 2. Thus, assuming a nonsymmetric function Φ' , that has only one pick at zero, it would not change our results from a qualitative point of view. For the sake of simplicity in exposition we assume that Φ' is symmetric around zero.

The limit difference of demand fractions. According to Proposition 1, we can approximate the trajectories of $(m_t^{(N)})$ to the trajectories of (m_t) when the number of consumers N is large. This implies that $\lim_{t\to\infty} m_t$ may be viewed as a time-stable difference of demand fractions when N is large; to be precise: if the difference of demand fractions starts from some $m_0^{(N)} = m_0$ and if we wait for a given relaxation time until we see that $m_t^{(N)}$ is more or less constant in time, then the value of $m_t^{(N)}$ will be close to

$$G(m_0;h) \stackrel{\text{def}}{=} \lim_{t \to \infty} g_t(m_0;h) \tag{15}$$

where $g_t(m_0; h)$ denotes the t - th iteration of mapping $m \mapsto 2\Phi(Jm - h) - 1$ starting from m_0 . We will therefore call $G(m_0; h)$

the limit difference of demand fractions



Figure 2: Generic shapes of the graph of $m \mapsto g(m;h) \stackrel{\text{def}}{=} 2\Phi(Jm-h) - 1$ for different values of $\Phi'(0), J$ and h. Case a: $\Phi'(0)^{-1} \ge 2J$. Cases b-f: $\Phi'(0)^{-1} < 2J$. Sub-cases: b: $h < -h_*$; c: $h = -h_*$; d: $-h_* < h < h_*$; e: $h = h_*$; f: $h > h_*$.



Figure 3: Case $\Phi'(0)^{-1} \ge 2J$. Generic shape of $h \mapsto \overline{M}(h)$ that represents the dependence on h of the unique stable equilibrium of $m_{t+1} = 2\Phi(Jm_t - h) - 1$.



Figure 4: Case $\Phi'(0)^{-1} < 2J$. Generic shapes of functions $h \mapsto M_+(h)$, $h \mapsto M(h)$ and $h \mapsto M_-(h)$ that represent the dependence on h of the equilibria of $m_{t+1} = 2\Phi(Jm_t - h) - 1$.

It should be clear that $G(m_0; h)$ corresponds to exactly one fix point of $m \mapsto 2\Phi(Jm - h) - 1$. Depending on m_0 , h, $\Phi(0)$ and J, the limit difference of demand fractions $G(m_0; h)$ will be either $\overline{M}(h)$, $M_-(h)$, M(h) or $M_+(h)$ as indicated in Figure 2.¹⁵

Heterogeneity and discontinuity of demands. The role of initial conditions. As suggested in Figure 2, the relationships between $\Phi'(0)^{-1}$ and J play an important role in determining if the limit difference of demand fractions $G(m_0; h)$ is discontinuous in h or not. If $\Phi'(0)^{-1} \geq 2J$, then $G(m_0; h) = \overline{M}(h)$ where $\overline{M}(h)$ is the unique global stable equilibrium of (m_t) , which is continuous and decreasing in h as shown in Figure 3. If $\Phi'(0)^{-1} < 2J$, then the limit difference of demand fractions $G(m_0; h)$ is no longer continuous in h, where the discontinuity point of $h \mapsto G(m_0; h)$ depends on the initial difference of demand fraction m_0 . To see this, fix the initial value m_0 and observe the limit $G(m_0; h) = \lim_{t\to\infty} m_t$ for different values of h. Suppose, for example, that $\Phi(0)^{-1}$, J and h are given as in d) of Figure 2. Assume initially $M(h) < m_0$. Increasing h moves the curve $h \mapsto 2(Jm-h)-1$ as well as the unstable equilibrium M(h) from the left to the right. As long as $M(h) < m_0$, it holds that $G(m_0; h) = M_+(h)$. As soon as M(h) exceeds m_0 , the limit difference of demand fractions $G(m_0; h) = m_+(h)$ to from $M_-(h)$. Figures 5, 6 and 7 show discontinuity points of $h \mapsto G(m_0; h)$ for $m_0 > M_*$, $m_0 < -M_*$ and $|m_0| \leq M_*$ respectively.

In order to interpret inequalities $\Phi'(0)^{-1} \geq 2J$ and $\Phi'(0)^{-1} < 2J$ that lead to two respective regimes of the limit difference of demand fractions (continuous and discontinuous) let Φ be parameterized as follows: $\Phi(x) = \Phi_1(x/\sigma)$, where Φ_1 is a cumulative probability distribution function that has finite variance and satisfies (5). For such a parametric Φ , the quantity $\Phi'(0)^{-1}$ is proportional to the standard deviation of Φ . For example, if $\Phi(x) = \Phi_1(x/\sigma)$, where Φ_1 is the cumulative probability distribution function of the standard normal distribution, then the standard deviation of Φ is $(2\pi)^{-1/2} * \Phi'(0)^{-1}$.

Supposing such a suitable parametric form for Φ , we can interpret $\Phi'(0)^{-1}$ as a measure for the heterogeneity of consumers' intrinsic preferences. Therefore, we will say that

 $\Phi'(0)^{-1}$ is the heterogeneity of consumers' intrinsic preferences.¹⁶

According to the interpretations of $\Phi(0)^{-1}$ and J we emphasize two regimes of demand: i) high heterogeneity regime where $\Phi'(0)^{-1} \ge 2J$ and ii) low heterogeneity regime where $\Phi'(0)^{-1} < 2J$.

5 Similarity of products, polarization and strategies

In the previous sections we explained the existence of two phases of demand under low heterogeneity of consumers' intrinsic preferences. In this section we will show that the

¹⁵Although $G(m_0; h)$ depends on parameters m_0 , h, $\Phi(0)$ and J, we suppressed $\Phi(0)$ and J in order to stress the dependence between $G(m_0; h)$ and $(m_0; h)$.

¹⁶It should be mentioned that this interpretation is valid for the case where Φ is parameterized by $\Phi(x) = \Phi_1(x/\sigma)$. However, in general there is no simple correspondence between $\Phi'(0)^{-1}$ and the standard deviation of Φ .



Figure 5: Graph of $h \mapsto G(m_0; h) \stackrel{\text{\tiny def}}{=} \lim_{t \to \infty} g_t(m_0; h)$ for $m_0 \in (M_*, 1)$.



Figure 6: Graph of $h \mapsto G(m_0; h) \stackrel{\text{\tiny def}}{=} lim_{t \to \infty} g_t(m_0; h)$ for $m_0 \in (-1, -M_*)$.



Figure 7: Graph of $h \mapsto G(m_0; h) \stackrel{\text{\tiny def}}{=} lim_{t \to \infty} g_t(m_0; h)$ for $m_0 \in [-M_*, M_*]$.

similarity of products will determine the low heterogeneity regime. This means that polarization of demand will be assured by proximity of products. This is a particulary interesting result if we contrast it with Becker's (1991) restaurant case, where demand polarization was observed in a duopoly of two similar product variants (the restaurants). In this section we will also analyze a dynamic game of price competition suggesting that demand will be polarized and locked at one specific player's side whenever products are sufficiently similar to each other.

We will also propose a game generalization where the product locations will be part of players' strategies. A Nash equilibrium of this game suggests that producers will locate their products close to each other (increasing their similarities) whenever the strength of social susceptibility among consumers is greater than a critical value. By contrast, when the strength of social susceptibility of consumers is smaller than this critical value one restores Hotelling's (1929) standard result according to which the distance between product variants is maximal.

Similarity of products and phase transition. Becker developed a model to explain why a duopoly of two similar restaurants (in prices and food quality) could be characterized by an excess demand of one and an excess supply of the other. His description of the under-demanded restaurant goes as follows: "...Almost directly across the street is another seafood restaurant with comparable food, slightly higher prices, and similar service and other amenities. Yet this restaurant has many empty seats most of the time..."

We claim that such a similarity of product variants, as observed in Becker's analysis, produces a low heterogeneity regime of consumers' intrinsic preferences, according to which demand polarization occurs.

In order to clarify the causality between similarity of product variants and low het-

erogeneity regime of consumers' intrinsic preferences, recall that $\theta^{(i)}$ can be viewed as a consumer's *i* intrinsic reservation price difference. That is, under the supposition that both establishments are equally demanded, consumer *i* will choose $E^{(+)}$ if $p(+) - p(-) < \theta^{(i)}$, and $E^{(-)}$ if $p(+) - p(-) > \theta^{(i)}$. Now, if both product variants are highly similar (ultimately identical), it is natural to suppose that the reservation price difference of each consumer *i* is close to zero, when both product variants are equally demanded, that is, $\theta^{(i)} \simeq 0$, $i \in C$. The rationale behind this is that nobody is willing to pay much more for good $E^{(-)}$ than for good $E^{(+)}$ – and vice-versa – when $E^{(-)}$ and $E^{(+)}$ are highly similar. This becomes obvious when $E^{(-)}$ and $E^{(+)}$ are identical.

Now, if random variables $\theta^{(i)} \simeq 0$, $i \in C$, are concentrated around zero, then they are also characterized by a low dispersion (low $\Phi(0)^{-1}$). Thus, if the two product variants are highly similar, it is natural to assume that low heterogeneity regime will prevail $(\Phi(0)^{-1} < 2J)$.

We stress that our model can be interpreted as a Hotelling's (1929) type model. From this point of view, the additional utilities $u_i(-)$ and $u_i(+)$ introduced in (1) may be viewed as decreasing functions of the distances between consumer i and the location of product variants $E^{(-)}$ and $E^{(+)}$ respectively. The smaller the distance between $E^{(-)}$ and $E^{(+)}$ (that is, the more similar are product variants), the lower will be the dispersion of consumers' intrinsic preferences $\theta^{(i)} = u_i(+) - u_i(-)$, $i \in C$. As soon as the distance between $E^{(-)}$ and $E^{(+)}$ is smaller than a critical distance, then the demand system goes into low heterogeneity regime. In order to see this in a concrete example, assume that consumers' addresses are random variable $l^{(i)}$, $i \in C$, uniformly distributed along a circle of circumference 1.¹⁷ Assume that $E^{(-)}$ and $E^{(+)}$ differ only in their spatial locations, $l^{(-)}$ and $l^{(+)}$, on the circle. Suppose that the transportation cost incurred by consumer i by visiting point $l^{(x)}$ – where product $E^{(x)}$ is delivered – is

$$\nu \cdot [d(l^{(i)}, l^{(x)})]^2$$

where ν is a non negative constant and $d(l^{(i)}, l^{(x)})$ is the shortest distance (geodesic) between $l^{(i)}$ and $l^{(x)}$ along the circle $(x \in \{-, +\})$.

Let us assume that the additional utility $u^{(i)}(x)$ of consumer *i* (given in (1)) corresponds to the negative value of consumer's *i* transportation cost incurred with consumption of $E^{(x)}$ ($x \in \{-,+\}$). Accordingly, the intrinsic preference of consumer *i* is

$$\theta^{(i)} = u^{(i)}(+) - u^{(i)}(-) = \left[-\nu \cdot \left[d(l^{(i)}, l^{(+)}) \right]^2 \right] - \left[-\nu \cdot \left[d(l^{(i)}, l^{(-)}) \right]^2 \right]$$
(16)

Given (16), we can derive the probability distribution of $\theta^{(i)}$. This is presented in the following proposition.

Proposition 2. Let $l^{(-)}$ and $l^{(+)}$ be fixed points (product locations) on a circle of circumference 1. Let $l^{(i)}$, $i \in C$, be independent and uniformly distributed random points along the circle (consumers' addresses). Let $d(l^{(i)}, l^{(-)})$, $d(l^{(i)}, l^{(+)})$ and d ($d = d(l^{(-)}, l^{(+)})$)

¹⁷The proposed circular address space is chosen for convenience in presentation. The same qualitative results can be achieved assuming that the address space is the unit interval.

denote the shortest distances (geodesics) between $l^{(i)}$ and $l^{(-)}$, between $l^{(i)}$ and $l^{(+)}$, and between $l^{(-)}$ and $l^{(+)}$ respectively. If $\theta^{(i)}$ is defined by (16), then $\theta^{(i)}$, $i \in C$, are independent and identically distributed random variable satisfying: $\theta^{(i)} = \theta - \xi^{(i)}$, $i \in C$, where

 $\theta = 0$ and $\xi^{(i)}$ is uniformly distributed on the interval $[-\delta, \delta]$, with $\delta = \nu d(1-d)$ (17)

According to (17), the cumulative distribution function Φ of $\xi^{(i)}$ is defined as follows: $\Phi(z)$ is identical to zero for $z \leq -\delta$, it is increasing and linear for $-\delta \leq z \leq \delta$, and identical to 1 for $z \geq \delta$.

The of Proposition 2 is presented in the Appendix.

Note that distance $d = d(l^{(-)}, l^{(+)})$ ranges between 0 and 1/2. This is because $l^{(-)}$ and $l^{(+)}$ are points on a circle of circumference 1. Accordingly, the dispersion of $\xi^{(i)}$'s, expressed by $\delta = \nu d(1 - d)$, is an increasing function of d ($0 \le d \le 1/2$) – the distance between product variants.

We recall that Φ was supposed to be differentiable (see (5)). Although Φ derived in Proposition 2 is not differentiable at $-\delta$ and δ , the *S*-shape of $m \mapsto 2\Phi(Jm-h) - 1$ allows us to derive analogous results about the existence and stability of the equilibria of $m_t = 2\Phi(Jm_{t-1} - h) - 1$. Figure 8 illustrates a case in which $\lim_{t\to\infty} m_t = 1$. In this case, $\delta < J$, $|h| < h_* \stackrel{def}{=} J - \delta$, and system $m_t = 2\Phi(Jm_{t-1} - h) - 1$ has two stable equilibria (low heterogeneity regime). If $\delta > J$, then system $m_t = 2\Phi(Jm_{t-1} - h) - 1$ has only one stable equilibrium (high heterogeneity regime).

Since $\delta = \nu d(1 - d)$, low heterogeneity regime is assured whenever $\nu d(1 - d) < J$, that is, whenever the distance d between product variants is sufficiently small. More specifically, for $J \leq \nu/4$, low heterogeneity regime is assured if

$$d = d(l^{(-)}, l^{(+)}) < (1 - \sqrt{1 - 4J/\nu})/2$$

If $J > \nu/4$, low heterogeneity regime is always assured regardless of the value of $d(l^{(-)}, l^{(+)})$.

Strategic interaction of producers. As demonstrated above, proximity of products leads to demand polarization. This means that demand tends to be polarized at one producer's side - supposed that prices remain fixed over time. If the under-demanded establishment, say $E^{(-)}$, does not change its price, it seems clear that the over-demanded one, $E^{(+)}$, will not raise its price beyond a discontinuity price which depends on the fixed price charged by $E^{(-)}$. This would explain the stability of demand polarization observed in Becker's restaurant case, provided that $E^{(-)}$ remains passive over time.

At this point the following question arises: why should the under-demanded establishment, $E^{(-)}$, remain passive over time? Finally, $E^{(-)}$ could set a low price and polarize demand at his side.

The above question suggests that price competition among similar producers leads to instability of market shares - polarization would change from one side to the other as a consequence of price competition. In what follows we show that this is not the



Figure 8: Generic shape of the graph of $m \mapsto g(m;h) \stackrel{\text{def}}{=} 2\Phi(Jm-h)-1$. If $m < (h-\delta)/J$, then g(m;h) = -1; If $(h-\delta)/J \le m \le (h+\delta)/J$, then $g(m;h) = (J/\delta)m - (h/\delta)$; If $m > (h+\delta)/J$, then g(m;h) = 1.

case when producers are informed about each other's costs and the strength of social interactions among consumers. We outline our arguments in terms of a dynamic game of price competition (the price-game).

We will also propose a game generalization in which product locations will belong to the strategy space of producers (the location-price-game). Based on this game generalization we will also explain that producers will locate their products close to each other when the strength of social interactions among consumers is greater than a critical value. To the contrary, producers will locate their products far away from each other, when the strength of social interactions among consumers is smaller than this critical value. Interestingly, for weak social interactions among consumers, one restores Hotelling's (1929) standard result that predicts maximal distance between competitors. We stress that exactly the opposite result can be expected when producers are supposed to deal with consumers that are strongly susceptible to one another's choices.

The price-game we will present should capture the essential mechanisms of price competition in the presence of social interactions of consumers. In order to get a simple model, we assume that players set prices over time aiming to achieve a sequence of demand equilibria over time. This point of view assumes that equilibria of demands are approached very quickly and that players have no control of demand dynamics out of their equilibria. More precisely, given an equilibrium of difference of demand fractions at time t - 1, say \bar{m}_{t-1} , players' prices $p_t(-)$ and $p_t(+)$ produce a new equilibrium \bar{m}_t at time t given by $\bar{m}_t = G(\bar{m}_{t-1}, h_t) = \lim_{\tau \to \infty} m_{\tau}$, where $m_0 = \bar{m}_{t-1}$ and $m_{\tau} = 2\Phi(Jm_{\tau-1} - h_t) - 1$ for $\tau > 0$. When it happens that \bar{m}_{t-1} is an equilibrium which is a repulsor, that is, when \bar{m}_{t-1} is a repulsor in respect to mapping $m \to 2\Phi(Jm - h_t) - 1$, then we will set $\bar{m}_t = G(\bar{m}_{t-1}, h_t + \zeta_t)$, where ζ_t is a random perturbation symmetrically distributed around zero. We formalize this game below.

The price-game. The game has two players and is played at times t = 1, 2, 3, ... At each time t = 1, 2, ..., players $E^{(-)}$ and $E^{(+)}$ choose simultaneously two respective prices satisfying $p_t(-) \ge c$ and $p_t(+) \ge c$.¹⁸ These prices are chosen based on the equilibrium of difference of demand fractions \bar{m}_{t-1} , that is,

$$p_t(x) = P_t^{(x)}(\bar{m}_{t-1}), \quad x \in \{-, +\}$$
(18)

where \bar{m}_0 is a game parameter.

Given the price choices at time t, $p_t(-)$ and $p_t(+)$, the resulting equilibrium of difference of demand fractions \bar{m}_t at time t is defined as follows:

$$\bar{m}_t = G(\bar{m}_{t-1}, h_t + \varepsilon_t), \quad h_t = p_t(+) - p_t(-) - \theta$$
 (19)

where $G(\bar{m}_{t-1}, h_t + \varepsilon_t) = \lim_{\tau \to \infty} m_{\tau}$ with $m_{\tau} = 2\Phi(Jm_{\tau-1} - h_t + \varepsilon_t) - 1$ for $\tau > 0$ and $m_0 = \bar{m}_{t-1}$. The random perturbation ε_t in (19) is defined as follows:

$$\varepsilon_t = \varepsilon_t(\bar{m}_{t-1}, h_t) = \begin{cases} \zeta_t & \text{if } \bar{m}_{t-1} \text{ is a repulsor at time } t \\ 0 & \text{otherwise} \end{cases}$$
(20)

In (20) we use the term "repulsor at time t" in the following sense: \bar{m}_{t-1} is a repulsor at time t if it solves the equation $m = 2\Phi(Jm - h_t) - 1$ and if it is a repulsor in respect to mapping $m \to 2\Phi(Jm - h_t) - 1$. Above, $\zeta_1, \zeta_2, \zeta_3, \ldots$ are independent normally distributed random variables with mean zero and positive variance (independent random perturbation symmetrically distributed around zero, where the value of ζ_t is not known at time t).

The pay-offs of players are

$$\sum_{t=1}^{\infty} \lambda^{t-1} \mathbb{E}\Big[\bar{N}^{(x)}(\bar{m}_t) \cdot \big[P_t^{(x)}(\bar{m}_{t-1}) - c\big]\Big], \quad x \in \{-, +\}$$
(21)

where:

- $P_t^{(x)}(\bar{m}_{t-1})$ and c, denote respectively the unit price and cost of $E^{(x)}$
- $\bar{N}^{(x)}(\bar{m}_t)$ is the resulting market share of $E^{(x)}$ at the equilibrium \bar{m}_t , that is,

$$\bar{N}^{(x)}(\bar{m}_t) = (1 + x \cdot \bar{m}_t)/2, \quad x \in \{-,+\}$$

• $\mathbb{E}(\cdot)$ is the mathematical expectation operator (due to the randomness of ζ_t 's)

¹⁸Prices are bonded below by the products' marginal costs - dumping prices are not allowed. This restriction is imposed to facilitate exposition. Analogous results can be derived assuming $p_t(-) \ge 0$ and $p_t(+) \ge 0$.

• λ is a discount factor, $0 < \lambda < 1$

The next proposition, Proposition 3, shows that in duopolies of interacting consumers demand polarization will occur and will be locked at one player's side whenever products are sufficiently close to each other, that is, whenever products are sufficiently similar to each other. To the contrary, when products are sufficiently distant from each other, Proposition 3 predicts that the market will be shared symmetrically among producers. More precisely, we will derive a subgame perfect Nash equilibrium (SPNE)¹⁹ of the pricegame that confirms this assertion. Proposition 3 assumes that θ and Φ are as derived in Proposition 2, that is: $\theta = 0$ and Φ is the cumulative probability distribution of the uniform distribution on the interval $[-\delta, \delta], \delta \geq 0$. Recall that " $\theta = 0$ " and the above parametric form of Φ were explicitly derived from the assumption that consumers and products are located along a circle of circumference 1, where $\delta = \nu d(1-d)$ with d denoting the shortest distance between products along the circle (Proposition 2).

We will also assume that $\bar{m}_0 = 0$. This is a natural assumption when products are supplied for the first time, and consumers have no information about the products' popularity - no bias in social influence. We will use this assumption when we introduce decisions about product locations in the strategy space of players.

Proposition 3. Consider the price-game defined above with pay-off functions (21). Assume that Φ is the cumulative distribution of the uniform distribution on the interval $[-\delta, \delta]$. Assume also that $\theta = 0 = \overline{m}_0$. Then for fixed $J \ge 0$ and fixed $\delta \ge 0$ $(\delta = \nu d(1-d), d = d(l^{(-)}, l^{(+)}))$, the following profile of price strategies (depending on δ) is a subgame perfect Nash Equilibrium of the price-game:

$$(P^{(-)}, P^{(+)}) = (P^{(-,\delta)}, P^{(+,\delta)})$$

where for $x \in \{-,+\}$ and $t \ge 1$:

$$P_t^{(x,\delta)} = \begin{cases} \begin{cases} J - \delta + c & \text{if } t > 1 \text{ and } \bar{m}_{t-1} = x \cdot 1 \\ c & \text{otherwise} \end{cases} & \text{if } \delta \leq J \\ \delta - J + c & \text{if } \delta > J \end{cases}$$
(22)

Furthermore the interaction of the above strategies leads to

$$Prob(\bar{m}_t = x, \ \forall t \ge 1) = \begin{cases} 1/2 & \text{if } |x| = 1 & \text{and} \quad \delta < J \\ 1 & \text{if } x = 0 & \text{and} \quad \delta \ge J \end{cases}$$
(23)

¹⁹Roughly speaking, a strategy profile $(P^{(-)}, P^{(+)})$ is a subgame perfect Nash equilibrium if it induces a Nash equilibrium in every subgame played from time t onward $(t \ge 1)$. For detailed descriptions of dynamic games and subgame perfect Nash equilibria the reader is referred to Fudenberg and Tirole (1991) and Vega-Redondo (2003).

It follows from (21), (22) and (23) that players' pay-offs in this Nash equilibrium are: $\pi^{(-)} = \pi^{(+)} = \pi(\delta)$, where

$$\pi(\delta) = \begin{cases} \frac{1}{2} \left[\frac{\lambda}{1-\lambda} (J-\delta) \right] & \text{if } \delta \le J \\ \frac{1}{2} \left[\frac{1}{1-\lambda} (\delta-J) \right] & \text{if } \delta > J \end{cases}$$
(24)

The proof of Proposition 3 is presented in the Appendix.

We stress that Nash equilibrium (22) depends on the (fixed) game parameters δ and J. In what follows we will discuss Proposition 3 in the following cases: $\delta < J$, $\delta = J$ and $\delta > J$. We will also generalize the price-game in the sense that δ will be determined by the players' product locations ($\delta = \nu d(1 - d)$, $d = d(l^{(-)}, l^{(+)})$). We will also derive some results related to this game generalization and discus it. For this we will adopt the same notation for prices and price strategies. To be more precise in respect to notations, we make the following remark.

About Notation. Formally, a strategy of player $E^{(x)}$ is a sequence of functions $P^{(x)} = \{P_t^{(x)}(\cdot)\}_{t=1}^{\infty}$, where $P_1^{(x)}(\cdot)$ is a constant function $(x \in \{-,+\})$. By setting $p_1(x) = P_1^{(x)}(\cdot)$, $x \in \{-,+\}$, and applying (18) and (19), the strategy profile $(P^{(+)}, P^{(-)})$ generates a unique two-dimensional price process $(p(+), p(-)) = \{p_t(+), p_t(-)\}_{t=1}^{\infty}$. For convenience in further expositions, we will refer componentwise to the strategy profile $(P^{(+)}, P^{(-)})$ as well to the generated price processes (p(+), p(+)) with the same notation $(P^{(+)}, P^{(-)})$.

We discuss now Nash equilibrium (22) when $\delta < J$. Since $\delta < J$, low heterogeneity regime is assured and polarization will occur. Accordingly, players will set their prices as low as possible (dumping prices not allowed) in order to try to polarize demand at time t = 1. After demand is polarized, the winner will raise his price up to the discontinuity price $J - \delta + c$ - assuming that the loser sets his price equal to c (lower bound for prices in this game). Note that the loser cannot improve his pay-off by setting a higher price at any subsequent decision time. Moreover, if the loser would set $c + \epsilon$ ($\epsilon > 0$) at a certain decision time t > 1, the best response of the winner at time t + 1 would be a price that is higher than $J - \delta + c$. But if the winner plays a price that is higher than $J - \delta + c$, the loser can set a relatively low price and become the winner from time t + 2 onward. The fact the loser plays a price equal to c reflects his behavior of permanently trying to become the winner - as soon as the current winner makes a mistake and exceeds his price over discontinuity price $J - \delta + c$, the loser will polarize demand and become the winner. Accordingly, the only subgame perfect Nash equilibrium (SPNE) where demand remains permanently polarized at one specific player's side is (22). It is also easy to see that (22)is the unique SPNE if J is sufficiently large, and prices are bonded above. Assertion (23) just says that demand will be polarized from time t = 1 onward at each player's side with probability 1/2.

The case $\delta = J$ can be viewed as a limit case of $\delta < J$ when $\delta \uparrow J$. Although demand polarization does not occurs, the players' discontinuity price are c, provided that their competitors' price are also c.

Let us now analyze Nash equilibrium (22) when $\delta > J$ (high heterogeneity regime). In this case, the price-game becomes a repeated game of price competition. This occurs because the equilibrium of difference of demand fractions \bar{m}_t depends only on prices $P_t^{(-)}$ and $P_t^{(+)}$ played at time t (not on \bar{m}_{t-1}). That is, \bar{m}_t equals the unique solution \bar{m} of

$$\bar{m} = 2\Phi \left(J\bar{m} + P_t^{(-)} - P_t^{(+)} \right) - 1$$
(25)

Consequently players' pay-offs at time $t, \pi_t^{(x)}, x \in \{-, +\}$, depend only on prices charged at that time. According to (21) and (25), the players' pay-offs at time t are:

$$\pi_t^{(x)} = N^{(x)}(\bar{m}) \left[P_t^{(x)} - c \right], \quad x \in \{-, +\}$$
(26)

where \bar{m} is the unique solution of (25).

Equation (26) implies that the price-game restricted to time t + 1 is exactly the same as the price-game restricted to time $t, t \ge 1$. Therefore, when $\delta > J$, a SPNE of the price-game is given by repetitions of the Nash equilibrium restricted to time t = 1. In the proof of Proposition 3 (see Appendix) we compute the indicated Nash equilibrium. It corresponds to the strategy profile (22) when $\delta > J$. In the proof of Proposition 3 we conclude also that the market will be shared symmetrically between $E^{(-)}$ and $E^{(+)}$ over time whenever $\delta > J$. This is asserted in (23) for $\delta > J$ ({ $\bar{m}_t = 0, \forall t \ge 1$ } occurs with probability 1).

Let us now analyse pay-off $\pi(\delta)$ presented in (24). Note that $\delta \to \pi(\delta)$ is decreasing for $\delta \leq J$ and increasing for $\delta > J$. Since δ is determined by products proximity d $(\delta = \nu d(1-d), d = d(l^{(-)}, l^{(+)}))$, it would be interesting to investigate the game outcomes in respect to locations, prices and resulting market shares when product locations $l^{(-)}$ and $l^{(+)}$ are chosen by the players.

In what follows we will define a second game in which players first locate their products and then play the price-game proposed before. In this game, product locations will determine the probability distribution Φ . As in the price-game defined before, we assume that Φ is the cumulative distribution function of the uniform distribution over $[-\delta, \delta]$, $\delta = \nu d(1 - d), d = d(l^{(-)}, l^{(+)})$, which results when products and consumers are located along a circle of circumference 1, and transportation costs are quadratic in the distances (geodesics) between consumers and products.

Location-price game. The game has two players $E^{(-)}$ and $E^{(+)}$, and is played at times t = 0, 1, 2, ... At time t = 0, players choose respective locations, $l^{(-)}$ and $l^{(+)}$, along a circle of circumference 1. From time t = 1 onward, players play the price-game defined before knowing that Φ is the cumulative distribution function of the uniform distribution over $[-\delta, \delta]$, where $\delta = \nu d(1 - d)$, $d = d(l^{(-)}, l^{(+)})$ (d is the shortest distance between $l^{(-)}$ and $l^{(+)}$ along the circle). The players' pay-offs are the same as in (21).

Let us consider the following strategy profile

$$[(l^{(-)}, P^{(-,\delta)}); \ (l^{(+)}, P^{(+,\delta)})], \quad \delta = \nu d(1-d), \ d = d(l^{(-)}, l^{(+)})$$
(27)

where $P^{(x,\delta)}$, $x \in \{-,+\}$ is the SPNE of the price-game defined in (22).

By applying backward induction to (27) we can compute the following Nash equilibrium of the location-price game:

$$[(l_*^{(-)}, P_*^{(-,\delta_*)}); \ (l_*^{(+)}, P_*^{(+,\delta_*)})]$$
(28)

where $\delta_* = \operatorname{argmax}_{\delta} \pi(\delta)$ (with $\pi(\delta)$ defined in (24)), and $l_*^{(x)}$, $x \in \{-,+\}$, satisfying $d(l_*^{(-)}, l_*^{(+)}) = d_*$ for $\nu d_*(1 - d_*) = \delta_*$.

Let us compute Nash equilibrium (28). For this, observe first that δ ranges between 0 and $\nu/4$. This is because $\delta = \nu d(1 - d)$, and d (the shortest distance between $l^{(-)}$ and $l^{(+)}$ along the circle of circumference 1) ranges between 0 and 1/2.

According to (24) $\pi(\delta)$ is strictly decreasing for $\delta \leq J$, and strictly increasing for $\delta > J$. Since δ ranges between 0 and $\nu/4$, we have $\delta_* = 0$ or $\delta_* = \nu/4$. Of particular interest to us is the critical value J_* that makes $\pi(0) = \pi(\nu/4)$ for $J = J_*$. Taking (24) into account, J_* must satisfy

$$\frac{1}{2} \left[\frac{\lambda}{1-\lambda} (J_* - \delta) \right] \bigg|_{\delta=0} = \frac{1}{2} \left[\frac{1}{1-\lambda} (\delta - J_*) \right] \bigg|_{\delta=\nu/4} \quad \Rightarrow \quad J_* = \frac{\nu/4}{\lambda+1}$$

Moreover it holds that

 $\pi(0) - \pi(\nu/4) > 0$ for $J > J_*$ and $\pi(0) - \pi(\nu/4) < 0$ for $J < J_*$

Since $d \to \delta$ ($\delta = \nu d(1 - d)$) is strictly increasing for $d \in [0, 1/2]$, it follows from the above that $d_* = 0$ (distance between products is minimal) if and only if $J > J_*$ and $d_* = 1/2$ (distance between products is maximal) if and only if $J < J_*$.

Let us summarize the above results in the following proposition:

Proposition 4. Consider the following strategy profile of the location-price game.

$$S = [(l^{(+)}, P^{(+,\delta)}); \ (l^{(-)}, P^{(-,\delta)})], \quad \delta = \nu d(1-d), \ d = d(l^{(-)}, l^{(+)})$$

where $P^{(x,\delta)}$, $x\{-,+\}$, is defined in (22). Set

$$J_* = \frac{\nu/4}{1+\lambda}$$

1. Suppose $J < J_*$. In this case, S is a subgame perfect Nash equilibrium (SPNE) if and only if $d(l^{(-)}, l^{(+)}) = 1/2$ (distance between $l^{(-)}$ and $l^{(+)}$ is maximal). In this Nash equilibrium the resulting prices and difference of demand fractions satisfy:

$$P_t^{(+)} = P_t^{(-)} = \nu/4 - J + c, \ \bar{m}_t = 0, \ t = 1, 2, \dots$$
(29)

The players' pay-off are

$$\pi^{(x)} = \frac{1}{2} \left[\frac{1}{1-\lambda} (\nu/4 - J) \right], \quad x \in \{-, +\}$$
(30)

2. Suppose $J > J_*$. In this case, S is a SPNE if and only if $d(l^{(-)}, l^{(+)}) = 0$ (distance between $l^{(-)}$ and $l^{(+)}$ is minimal). In this Nash equilibrium the resulting prices and difference of demand fractions satisfy:

$$P_1^{(+)} = P_1^{(+)} = c,$$

$$P_t^{(x)} = J + c, \quad P_t^{(-x)} = c, \quad \bar{m}_1^{(x)} = x1, \quad \bar{m}_t = x1, \quad t = 2, 3, \dots$$
(31)

where the second line of (31) occurs with probability 1/2 for each fixed $x \in \{-,+\}$. The players' pay-off are

$$\pi^{(x)} = \frac{1}{2} \left[\frac{\lambda J}{1 - \lambda} \right], \quad x \in \{-, +\}$$
(32)

3. Suppose $J = J_*$. In this case, S is a SPNE if and only if $d(l^{(-)}, l^{(+)}) = 1/2$ or $d(l^{(-)}, l^{(+)}) = 0$ (distance between $l^{(-)}$ and $l^{(+)}$ is either maximal or minimal). If $d(l^{(-)}, l^{(+)}) = 1/2$, then S satisfy (29) and (30). If $d(l^{(-)}, l^{(+)}) = 0$, then S satisfy (31) and (32).

Proposition 4 shows that producers tend to come close to each other when J is sufficiently large, that is, when consumers are sufficiently susceptible to one another's choices. In this case, producers get more market power by producing a social product differentiation which happens when competing products are similar. This may explain the geographic concentration of bars, restaurants, night clubs under the circumstances that some of them are poorly visited. Since consumers are strongly susceptible to one another's choices, producers come close to each other and try to polarize demand at a first moment. After polarization occurs, some of them sustain a relatively high level of demand compared to their prices while others remain poorly visited in spite of their modest prices. Although frustrations are unavoidable, on average, it may be more advantageous for all players to come close to each other in order to exploit $\delta - J > 0$ (making δ small) than to be distant from each other in order to exploit $\delta - J > 0$ (making δ large). This is the case when δ is bonded above by a constant $\nu/4$, and J is sufficiently large (even when $J < \nu/4$).

By contrast, when J is small, that is, when consumers are weakly susceptible to one another's choices, one restores Hotelling's (1929) standard result, according to which producers locate their products faraway from each other. In this case, market players get their market power from the proximity to their potential costumers. This seems to be a natural assumption for producers (like gas stations, pharmacies, etc.) whose consumers are not primarily susceptible to one another's choices, but to the accessability of products.

The above examples relates to geographic locations of products and consumers. The same arguments hold also for product differentiations in the context of a space of product characteristics (see Anderson et al. (1992)).

6 Closing remarks

In this paper we present a stochastic model of heterogeneous interacting consumers deciding between two product variants. Our model predicts that interactions among consumers will lead to market polarization whenever products become sufficiently similar to each other. This model result extends Becker's (1991) explanation about the demand polarization of two similar restaurants across from each other competing for consumers.

We consider a mix of two types of consumers' heterogeneity: fixed heterogeneity and varying heterogeneity over time. We analyze a model parametrization, according to which, a fraction of habitual consumers (model parameter between 0 and 1) are permanently replaced by new consumers "refreshing the population of consumers" over time. A general model result shows that the resulting stochastic share of decisions converges to a specific dynamical system, regardless of the particular fraction of exchanging consumers under consideration. This convergence is proved in the case when the number of habitual consumers goes to infinity. This limit result is applied to deduce an interesting relationship between the fraction of exchanging agents and the process of consumers' decisions in the case when the number of habitual consumers is finite - instead of infinite. If this fraction is sufficiently small (not necessarily equal to zero) then the process is non-ergodic, otherwise it is ergodic. We discus an application of this process property and show how it can be used to drive an undesired state of decisions into a desired one.

We propose a dynamic game of price competition of two similar product variants in which consumers are susceptible to one another's choices of products. In this game, producers are aware of tree things: i) their product locations, ii) each other's product costs and ii) the strength of social interactions among consumers. We analyze this game and show that, in a Nash equilibrium, demand will be polarized and locked at one player's side whenever products are sufficiently close to each other. The game predicts the opposite result, that is, that market is shared symmetrically among competitors, whenever products are sufficiently distant from each other. We also analyze a game generalization in which product locations were considered in the strategy space of players. Interestingly, an equilibrium analysis of this game shows that players will locate their products close to each other whenever the strength of social interactions among consumers is larger than a critical value. By contrast, if the strength of social interactions among consumers is smaller than this critical value, then the game outcome restores Hotelling's (1929) standard result that predicts maximal distance of products.

The model results indicate that social interactions among consumers play a key role in determining the way producers interact strategically with each other, implying in quite different outcomes in respect to locations, prices and market shares.

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7 Appendix

Proof of Proposition 1. Our goal is to prove (33) as stated below

$$\forall t \ge 0: \quad \lim_{N \to \infty} \left| m_t^{(N)} - m_t \right| = 0, \quad \text{almost surely.}$$
(33)

In order to prove (33), we first prove the following convergence:

$$\forall t \ge 0: \quad \lim_{N \to \infty} \left| m_t^{(N)} - g(m_{t-1}^{(N)}) \right| = 0, \quad \text{almost surely,}$$
(34)

where g is defined by $g(m) = 2\Phi(Jm - h) - 1$.

Assertion (33) will then follow from (34) by induction on t.

Proof of (34). For the whole proof of (34) we will fix an arbitrary value for t.

Let us define

$$\mathbb{I}_{\{\xi^{(i)} \le x\}} \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if} & \xi^{(i)} \le x \\ 0, & \text{otherwise} \end{cases} \quad x \in \mathbb{R}, \quad i = 1, 2, \dots$$
(35)

and (for fixed t)

$$\Phi_N(x) \stackrel{\text{\tiny def}}{=} \frac{1}{N} \sum_{i \in C_t^{(N)}} \mathbb{I}_{\{\xi^{(i)} \le x\}}$$

$$(36)$$

Since for each $x \in \mathbb{R}$ and $N \geq 1$, random variables $\mathbb{I}_{\{\xi^{(i)} \leq x\}}$, $i \in C_t^{(N)}$ are independent Bernoulli random variables with expected value $\Phi(x)$ ($\Phi(x) = P(\xi^{(i)} \leq x)$), we conclude (using $|\mathbb{I}_{\{\xi^{(i)} \leq x\}}| \leq 1$ and Borel-Cantelli's Lemma) that

$$\forall x: \Phi_N(x) \to \Phi(x), \text{ almost surely as } N \to \infty$$
 (37)

Applying the argument, that proves Glivenko-Cantelli's theorem, 20 to (37) we deduce:

$$\lim_{N \to \infty} \sup_{x} \left| \Phi_{N}(x) - \Phi(x) \right| = 0, \quad \text{almost surely}$$
(38)

The uniform convergence asserted in (38) will be used below to prove (34).

We now observe that $m_t^{(N)} = 2[N_t(+1)/N] - 1$, where we recall that $N_t(+1)$ is the number of consumers that choose +1 at time t. This follows from (8) and the following relation: $N_t(-1) = N - N_t(+1)$. Using " $m_t^{(N)} = 2[N_t(+1)/N] - 1$ ", steps 1)-3) of the construction of process $(m_t^{(N)})_{t\geq 0}$, described in Section 3, and the notations introduced in (35) and (36), we can write

$$m_t^{(N)} = 2\Phi_N \left(J m_{t-1}^{(N)} - h \right) - 1 \tag{39}$$

²⁰Durrett (1995), p. 59-60.

Now, (39) and the definition of mapping $m \mapsto g(m) = 2\Phi(Jm - h) - 1$ imply that

The inequality in (40) and the convergence in (38) imply an almost certain convergence (34).

We are now in position to prove (33) by induction on t.

Proof of (33).

The induction basis (t = 0). For t = 0, the result is immediate, since " $m_0^{(N)} = m_0$, $\forall N \ge 1$ " is an assumption of the proposition.

The induction step $(t-1 \frown t)$. Suppose now that (33) holds for an arbitrary $t-1 \ge 0$, that is, $m_{t-1}^{(N)}$ converges almost surely to the constant value m_{t-1} (when $N \to \infty$). Since g is continuous, $g(m_{t-1}^{(N)})$ converges almost surely to $g(m_{t-1})$. Since $g(m_{t-1}) = m_t$, we conclude that $g(m_{t-1}^{(N)}) - m_t$ converges almost surely to zero.

Now, since according to (34), $m_t^{(N)} - g(m_{t-1}^{(N)})$ converges almost surely to zero, we can apply the triangular inequality and deduce the result for t:

$$|m_t^{(N)} - m_t| \le \le |m_t^{(N)} - g(m_{t-1}^{(N)})| + |g(m_{t-1}^{(N)}) - m_t| \to 0$$
(41)

This shows (33) and completes the proof of Proposition 1.

Proof of Proposition 2. We aim to find the probability distribution of $\theta^{(i)}$ (for $l^{(-)}$ and $l^{(+)}$ fixed). Since $l^{(-)}$, $l^{(+)}$ and $l^{(i)}$ are points along the circle, the only relevant information for computing the probability distribution of $\theta^{(i)}$ is distance $d = d(l^{(-)}, l^{(+)})$ (not the exact locations $l^{(-)}$ and $l^{(+)}$). For computing the probability distribution of $\theta^{(i)}$, we normalize $l^{(-)} = -d/2$ and $l^{(+)} = d/2$ ($d \le 1/2$), and adopt the convention that points (addresses) along the circle vary from -1/2 to 1/2, with -1/2 and 1/2 being assigned to the same point. Under this convention, (16) reduces to:

$$\theta^{(i)} = f(l^{(i)}) = \begin{cases} -2\nu(1-d)l^{(i)} - \nu(1-d) & \text{if } -1/2 \le l^{(i)} < -(1-d)/2 \\ 2\nu dl^{(i)} & \text{if } |l^{(i)}| \le (1-d)/2 \\ -2\nu(1-d)l^{(i)} + \nu(1-d) & \text{if } (1-d)/2 < l^{(i)} \le 1/2 \end{cases}$$
(42)

Now, since the random location $l^{(i)}$ is uniformly distributed over [-1/2, 1/2], the cumulative distribution function of $\theta^{(i)}$ can be computed by inspecting function $f(l^{(i)})$ defined in (42). In fact, by inspecting the graph of function f, it is easy to see that $\theta^{(i)}$ is uniformly distributed on the interval $[-\nu d(1-d), \nu d(1-d)]$. Since the interval $[-\nu d(1-d), \nu d(1-d)]$ is symmetric around zero, we can write $\theta^{(i)} = \theta - \xi^{(i)}$ and assume (17).

Proof of Proposition 3. We divide the proof into tree cases: $\delta < J$, $\delta = J$ and $\delta > J$.

Case $\delta < J$ (low heterogeneity regime).

Let us first prove (23). Due to (22) we have $P_1^{(-)} = P_1^{(+)} = c$. Hence, $h_1 = 0$. Since $J > \delta$ (low heterogeneity regime), \bar{m}_0 ($\bar{m}_0 = 0$) is a repulsor at time 1. Due to the randomness of ζ_1 , we have: $\bar{m}_1 = 1$ with probability 1/2 and $\bar{m}_1 = -1$ with probability 1/2. Given that $m_1 = 1$, we will argue that $\bar{m}_t = 1, \forall t > 1$ (analogously, we can conclude that $\bar{m}_t = -1, \forall t > 1$, given that $m_1 = -1$).²¹

Suppose $\bar{m}_1 = 1$. Due to (22), players would play $P_2^{(-)} = c$ and $P_2^{(+)} = J - \delta + c$ at time t = 2. Since $\bar{m}_1 = 1$ and $h_2 = P_2^{(+)} - P_2^{(-)} = J - \delta$, equilibrium \bar{m}_1 is not repulsor at time t = 2. Accordingly, $\bar{m}_2 = G(\bar{m}_1, h_2) = G(1, J - \delta) = 1$. Since $\bar{m}_2 = 1$, (due to (22)) players would play again $P_3^{(-)} = c$ and $P_3^{(+)} = J - \delta + c$ at time t = 3. By induction on t, we conclude: $\bar{m}_t = 1, \forall t \ge 1$.

Assertion (24) follows immediately from (21), (22) and (23) in case $\delta < J$.

Let us now verify that (22) is a subgame perfect Nash equilibrium (SPNE) of the price-game when $\delta < J$. Suppose it is not. We will derive a contradiction. If (22) is not a SPNE, there would be a time t_* where a player, say $E^{(+)}$, would get a higher pay-off from time t_* onward if he deviates from (22) from time t_* on. Suppose initially this time is $t_* = 1$. If $E^{(+)}$ sets $P_1^{(+)} > c$ (and $P_1^{(-)} = c$), we will get $h_1 > 0$. Since $h_1 > 0$ and $\bar{m}_0 = 0$, it follows that \bar{m}_0 is not a repulsor at time 1. Thus $m_1 = G(\bar{m}_0, h_1) = -1$. Now, for any price choice $P_2^{(+)}$ with $P_2^{(+)} \ge c$, it holds $h_2 = P_2^{(+)} - P_2^{(-)} \ge 0$ and \bar{m}_1 ($\bar{m}_1 = -1$) is not a repulsor at time 2. Accordingly, $\bar{m}_2 = G(\bar{m}_1, h_2) = G(-1, h_2) = -1$. Repeating this argument we conclude (by induction on t) that $\bar{m}_t = -1$, $\forall t \ge 1$. The latter assertion implies zero pay-off for $E^{(+)}$ and shows that $E^{(+)}$ can not optimally deviate from (22) at time $t_* = 1$.

Suppose there is a time $t_* > 1$ in which player $E^{(+)}$ could optimally deviate from (22) improving his pay-off from time t_* onward. If $\bar{m}_1 = -1$, there is nothing that $E^{(+)}$ could do to avoid a zero profit, that is, demand would be polarized and locked at player's $E^{(-)}$ side, and the pay-off of $E^{(+)}$ would be zero, regardless of the price choices of $E^{(+)}$ at times $t_* + 1, t_* + 2, t_* + 3, \ldots$

Let us assume $\bar{m}_1 = 1$. If there is such a time $t_* > 1$, there will be a deviating strategy $\tilde{P}^{(+)}$ that will result in a larger pay-off for $E^{(+)}$ from time t_* onward than strategy $P^{(+)}$ does. If the deviating strategy is always bonded above by strategy $P^{(+)}$, that is, if $\tilde{P}_{\tau}^{(+)} \leq P_{\tau}^{(+)}$ for $\tau = t_*, t_* + 1, \ldots$, then it should be clear that the resulting pay-off under $\tilde{P}_{\tau}^{(+)} > P_{\tau_*}^{(+)}$ at a first time τ_* with $1 < t_* \leq \tau_* < \infty$ and $\tilde{P}_{\tau}^{(+)} \leq P_{\tau}^{(+)}$ for all $\tau = 1, 2 \ldots \tau_*$. Then, given $\bar{m}_1 = 1$, the resulting pay-off under $\tilde{P}^{(+)}$ from time t_* onward (denoted below

²¹From assertions i) " $\bar{m}_t = 1, \forall t > 1$, given $\bar{m}_1 = 1$ " and ii) " $\bar{m}_t = -1, \forall t > 1$, given $\bar{m}_1 = -1$ ", it follows that $\{\bar{m}_t = 1, \forall t \ge 1\}$ and $\{\bar{m}_t = -1, \forall t \ge 1\}$ occur with probability 1/2. This proves (23) when $\delta < J$.

by $\tilde{\pi}^{(+)}(t \ge t_* | \bar{m}_1 = 1))$ can be estimated as follows:

$$\tilde{\pi}^{(+)}(t \ge t_* | \bar{m}_1 = 1) \le \sum_{\substack{\tau = t_* \\ \tau = t_*}}^{\tau_* - 1} \lambda^{\tau - t_*} (J - \delta) + \sum_{\substack{\tau = t_* \\ \tau = t_*}}^{\infty} 0 \\ < \sum_{\substack{\tau = t_* \\ \tau = t_*}}^{\infty} \lambda^{\tau - t_*} (J - \delta) = \pi^{(+)} (t \ge t_* | \bar{m}_1 = 1)$$
(43)

where $\pi^{(+)}(t \ge t_* | \bar{m}_1 = 1)$ denotes the pay-off of $E^{(+)}$ under $\tilde{P}^{(+)}$ from time t_* onward, given $m_1 = 1$. Condition (43) contradicts the fact that $E^{(+)}$ could optimally deviate from (22) at some time $t_* > 1$, given $m_1 = 1$.

Case $\delta = J$.

Let us first prove (23). Due to (22) we have $P_1^{(-)} = P_1^{(+)} = c$ and $h_1 = 0$. Since $\delta = J$, equilibrium \bar{m}_0 ($\bar{m}_0 = 0$) is not a repulsor at time t = 1. Accordingly, $\bar{m}_1 = G(\bar{m}_0, h_1) = G(0, 0) = 0$. Due to (22) we have again $P_2^{(-)} = P_2^{(+)} = c$ and $h_2 = 0$. By induction on t, it follows that $\bar{m}_t = 0$, $\forall t \ge 1$. This proves (23) when $\delta = J$. Assertion (24) follows immediately from (21), (22) and (23) in case $\delta = J$ (in this case the players' pay-off are zero). The proof that (22) is a SPNE (when $\delta = J$) is omitted. It is analogous to the corresponding proof in case $\delta < J$.

Case $\delta > J$ (high heterogeneity regime).

Let us first prove (23). Due to (22) we have $P_t^{(-)} = P_t^{(+)} = \delta - J + c$ and $h_t = 0$, $\forall t \ge 1$. Since $\delta > J$, equilibrium \bar{m}_0 ($\bar{m}_0 = 0$) is not a repulsor at time t = 1. Accordingly, $\bar{m}_1 = G(\bar{m}_0, h_1) = G(0, 0) = 0$. By induction on t, it follows that $\bar{m}_t = 0$, $\forall t \ge 1$. This proves (23) when $\delta \ge J$.

Assertion (24) follows immediately from (21), (22) and (23) in case $\delta > J$.

Let us now verify that (22) is a subgame perfect Nash equilibrium (SPNE) of the price-game when $\delta > J$. In this case $(\delta > J)$, the price-game becomes a repeated game of price competition. This occurs because the equilibrium of difference of demand fractions \bar{m}_t depends only on prices $P_t^{(-)}$ and $P_t^{(+)}$ played at time t (not on \bar{m}_{t-1}). That is, \bar{m}_t equals the unique solution \bar{m} of $\bar{m} = 2\Phi(J\bar{m} + P_t^{(-)} - P_t^{(+)}) - 1$. This means that the price-game restricted to time t + 1 is exactly the same as the price-game restricted to time $t, t \geq 1$. Accordingly, a SPNE of the price-game is given by repetitions of the Nash equilibrium of the price-game restricted to time t = 1. In what follows we show that $P_1^{(-)} = P_1^{(+)} = \delta - J + c$ is a Nash equilibrium of the

In what follows we show that $P_1^{(-)} = P_1^{(+)} = \delta - J + c$ is a Nash equilibrium of the price-game restricted to t = 1. For this purpose we first compute the equilibrium \bar{m}_1 when prices $P_1^{(-)}$ and $P_1^{(+)}$ are fixed (in the computation below, we suppress the time index from the variables involved).

Given fixed prices $P^{(-)}$ and $P^{(+)}$, the only (glabal stable) equilibrium of difference of demand fractions \bar{m} must satisfy:

$$\bar{m} = 2\Phi(J\bar{m} + P^{(-)} - P^{(+)}) - 1$$

Since Φ is the cumulative distribution function of the uniform distribution over $[-\delta, \delta]$, we conclude:

$$\bar{m} = \begin{cases} 1 & \text{if } \frac{P^{(-)} - P^{(+)}}{\delta - J} > 1 \\ \frac{P^{(-)} - P^{(+)}}{\delta - J} & \text{if } |\frac{P^{(-)} - P^{(+)}}{\delta - J}| \leq 1 \\ -1 & \text{if } \frac{P^{(-)} - P^{(+)}}{\delta - J} < -1 \end{cases}$$
(44)

Now, according to (21), the pay-off of $E^{(x)}$ at time t = 1 is

$$\pi_1^{(x)} = [(1 + x\bar{m})/2](P^{(x)} - c), \quad x \in \{-, +\}$$
(45)

From (44) and (45) we conclude:

$$\pi_1^{(x)} = I_u \left(\frac{P^{(-x)} - P^{(x)} + 1}{2(\delta - J)}\right) (P^{(x)} - c), \quad x \in \{-, +\}$$
(46)

where I_u denotes the following function defined on [0,1]: $I_u(y) = 0$ for y < 0, $I_u(y) = y$ for $y \in [0,1]$, and $I_u(y) = 1$ for y > 1.

Due to the symmetry of (46), it is easy to compute the Nash equilibrium in (46). It is given by:

$$P^{(+)} = P^{(-)} = \delta - J + c \quad \Box$$